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Symmetry reductions for the Tzitzeica curve equation

Nicoleta Bîlă *

Abstract

The purpose of the paper is to analyze the *Tzitzeica curve equation* from the point of view of symmetry analysis theory. The Tzitzeica curve equation is a nonlinear ordinary differential equation that arises from differential geometry that may be regarded as a differential equation in three unknown functions, namely, the functions that give the parametrical representation of the curve. The extended classical symmetries, the generalized equivalence transformations and the equivalence transformations related to this nonlinear ordinary differential equation are determined.

1 Introduction

One of the most prestigious Romanian mathematicians, Gheorghe Tzitzeica (1873-1939), introduced a type of curves that carry today his name. These curves have the property that they are centro-affine invariant which means that the image of a Tzitzeica curve through a centro-affine transformation is also a Tzitzeica curve. A *Tzitzeica curve* is a spatial curve for which the ratio of its torsion τ and the square of the distance d from the origin to the osculating plane at an arbitrary point of the curve is constant, i.e.,

$$\frac{\tau}{d^2} = \alpha, \tag{1}$$

where $\alpha \neq 0$ is a real constant. Interestingly, the nonlinear ordinary differential equation (ODE) that may be obtained from (1), namely (4), has not

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been studied deeply so far and, hence, only a few exact solutions are known (see, for instance, [1] and [6]).

On the other hand, symmetry analysis is one of the most powerful techniques that can be used to analyze nonlinear differential equations. Due to his research work on continuous groups of transformations (known today as *Lie groups of transformations*) and their applications to ODEs and to partial differential equations (PDEs), Sophus Lie (1842–1899) may be considered the founder of the symmetry analysis theory. Almost all important differential equations arising in mathematical physics have been studied from the point of view of symmetry analysis theory. Therefore, the literature on this topic is impressive (see, for example, [2], [7], [11], and [12]).

The aim of this paper is to determine specific symmetry reductions for the Tzitzeica curve equation. More exactly, its associated extended classical symmetries, generalized equivalence transformations and equivalence transformations are studied. Recently, N. Bîlă and Niesen [3]) introduced a new method for finding two new classes of symmetry reductions that may be applied to differential equations with arbitrary functions (or parameters) involving the model's independent variables (see also [4]). These new groups of transformations are called *extended classical symmetries* and *extended non-classical symmetries*. Since the nonclassical method may not be applied for ODEs, the extended nonclassical symmetries will not be studied in the case of the Tzitzeica curve equation (4). In this paper, we will determine the extended classical symmetries for (4) and, next, we will apply the method presented in [4] for finding the generalized equivalence transformations and the equivalence transformations related to this equation.

The structure of the paper is as follows. In Section 2, the Tzitzeica curve equation is described. In Section 3, the new classes of symmetry reductions for the Tzitzeica curve equation are determined. The conclusions of this work are presented in Section 4.

2 The Tzitzeica curve equation

Let us consider a curve defined parametrically by

$$\mathbf{r}(t) = (x(t), y(t), z(t)), \quad (2)$$

where $t \in I \subset \mathbf{R}$ is the curve parameter. Assume that (2) has nonzero curvature k . The torsion of the curve is defined as

$$\tau(t) = \frac{(\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t))}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|^2}$$

where here the primes denote the derivatives with respect to t , the vector $\mathbf{r}' \times \mathbf{r}''$ is the cross product of the tangent vector \mathbf{r}' and the acceleration vector \mathbf{r}'' , $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|$ is the magnitude of $\mathbf{r}' \times \mathbf{r}''$, and

$$(\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t)) = \begin{vmatrix} x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \\ x'''(t) & y'''(t) & z'''(t) \end{vmatrix}$$

is the mixed product of vectors \mathbf{r}' , \mathbf{r}'' , and \mathbf{r}''' (see, for instance, [13], page 48). We assume that (2) has nonzero torsion at each point on the curve. Next, we consider the equation of the osculating plane

$$\begin{vmatrix} x - x(t) & y - y(t) & z - z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix} = 0.$$

The osculating plane is generated by the unit tangent vector $\mathbf{T}(t)$ and the unit normal vector $\mathbf{N}(t)$ at each point of the curve or, equivalently, by the tangent vector $\mathbf{r}'(t)$ and the acceleration vector $\mathbf{r}''(t)$. Next, the distance from the origin to the osculating plane of the curve is

$$d^2 = \frac{1}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|^2} \begin{vmatrix} x(t) & y(t) & z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix}^2.$$

The substitution of τ and d^2 into the condition (1) yields the equation

$$\begin{vmatrix} x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \\ x'''(t) & y'''(t) & z'''(t) \end{vmatrix} = \alpha \begin{vmatrix} x(t) & y(t) & z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix}^2, \quad (3)$$

which may also be written as

$$az''' - a'z'' + bz' = \alpha (cz'' - c'z' + az)^2, \quad (4)$$

where

$$a = x'y'' - x''y', \quad b = x''y''' - x'''y'', \quad \text{and} \quad c = xy' - x'y$$

are functions of the curve parameter t . The equation (4) will be called the *Tzitzeica curve equation*.

Proposition 1. *The space curve (2) is a Tzitzeica curve if and only if the functions x , y , and z are solutions to the nonlinear ODE (4).*

3 Symmetry reductions associated with the Tzitzeica curve equation

3.1 Extended classical symmetries

Consider the following ODE system

$$\mathcal{F}_\nu(t, u^{(n)}) = 0, \quad \nu = 1, \dots, l, \quad (5)$$

where $V \subset \mathbf{R}$ is the space of the independent variable t and $U \subset \mathbf{R}^q$ is the space of the dependent variables $u = (u^1, \dots, u^q)$. By definition, the *classical symmetries* related to the ODE system (5) are (local) Lie groups of transformations of the form

$$T_\varepsilon : \begin{cases} \tilde{t} = f(t, u; \varepsilon) \\ \tilde{u} = g(t, u; \varepsilon), \end{cases}$$

where ε is the group parameter, that act on an open set $D \subset V \times U$ of the space of the independent variable t and the space of the dependent variables u of the equation with the property that they leave the system invariant, i.e.,

$$\mathcal{F}_\nu(\tilde{t}, \tilde{u}^{(n)}) = 0, \quad \nu = 1, \dots, l.$$

The main advantage of finding classical symmetries related to an ODE system is the reduction of the order of the system: if the ODE system (5) admits a Lie group of transformations, then the order of the system can be reduced by one. The classical symmetries related to an ODE system are found by solving an over-determined linear PDE system called the *determining equations of the symmetry group*. There are a few software packages designed to find the classical symmetries related to a ODE or PDE system. For the symbolic

manipulation program MAPLE, one can use the package DESOLV (authors Carminati and Vu [5]).

In the case of the Tzitzeica curve equation (4), the unknown function is z and the arbitrary functions are x and y . In fact, any of the functions x , y , and z may be chosen to be the unknown function. Consider the one-parameter Lie group of transformations defined by

$$T_\varepsilon : \begin{cases} \tilde{t} = f(t, z; \varepsilon) = t + \varepsilon\xi(t, z) + O(\varepsilon^2) \\ \tilde{z} = g(t, z; \varepsilon) = z + \varepsilon\eta(t, z) + O(\varepsilon^2) \end{cases}$$

that acts on an open set of the space of the independent variable t and dependent variables x, y, z of the Tziteica curve equation (4) whose infinitesimal generator, the so-called the *symmetry operator*, is given by

$$X = \xi(t, z)\partial_t + \eta(t, z)\partial_z. \quad (6)$$

In this case, the output of the DESOLV routine `gendef` is the overdetermined system of the determining equations of the symmetry group and this depends on the arbitrary functions x and y . Therefore, the infinitesimal generators ξ and η are more difficult to be found.

For ODE systems depending on arbitrary functions, one may study another class of symmetry reductions, called *extended classical symmetries*. According to [3], in this case, the system's arbitrary functions are also regarded as dependent variables. Thus, the original system becomes an ODE system in a larger number of dependent functions. Consider the ODE system

$$\mathcal{G}_\nu(t, \phi^{(m)}, u^{(n)}) = 0, \quad \nu = 1, \dots, l \quad (7)$$

which depends on the arbitrary functions $\phi(t) = (\phi^1(t), \dots, \phi^r(t))$. Similar to the case of classical symmetries, the *extended classical symmetries* are defined as (local) Lie groups of transformations $(T_\varepsilon)_{\varepsilon \in I}$ acting on an open set $D \subset V \times U \times W$ of the space of the independent variable t , the space of the original dependent variables u , and the space of the arbitrary functions (or the space of the new dependent variables) ϕ of the system, that is

$$S_\varepsilon : \begin{cases} \tilde{t} = f(t, u, \phi; \varepsilon) \\ \tilde{u} = g(t, u, \phi; \varepsilon) \\ \tilde{\phi} = h(t, u, \phi; \varepsilon), \end{cases}$$

where ε is the group parameter, with the property that they leave the original ODE system invariant, i.e.,

$$\mathcal{G}_\nu(\tilde{t}, \tilde{\phi}^{(m)}, \tilde{u}^{(n)}) = 0, \quad \nu = 1, \dots, l.$$

To determine the extended classical symmetries related to the Tzitzeica curve equation (4), we include the arbitrary functions x and y in the space of the dependent variables, and, consequently, the equation (4) becomes an ODE in the unknown functions x , y , and z . Let

$$Y = \xi(t, x, y, z)\partial_t + \eta_1(t, x, y, z)\partial_x + \eta_2(t, x, y, z)\partial_y + \eta_3(t, x, y, z)\partial_z$$

be the *extended classical symmetry operator* related to $(S_\varepsilon)_\varepsilon$. Next, the operator Y is prolonged to the space of the derivatives of the functions x , y , and z up to the order three (note that the Tzitzeica curve equation contains third order derivatives of x , y , and z). Denote by $\text{pr}^{(3)}Y$ the third order prolongation of the extended symmetry operator of Y (for more details, see, for instance, [11]). Next, we apply the criterion for infinitesimal invariance that states that

$$\text{pr}^{(3)}(Y)|_{\mathcal{F}=0} = 0$$

where $\mathcal{F} = 0$ is the Tzitzeica curve equation (4). To determine the extended classical symmetries, one may use the MAPLE package DESOLV or the subroutine presented in [4]. It follows

Proposition 2. *The extended classical symmetries related to the Tzitzeica curve equation (4) are given by*

$$T_\varepsilon : \begin{cases} \tilde{t} = f(t, x, y, z; a_0) \\ \tilde{x} = a_{11}x + a_{12}y + a_{13}z \\ \tilde{y} = a_{21}x + a_{22}y + a_{23}z \\ \tilde{z} = a_{31}x + a_{32}y + a_{33}z, \end{cases}$$

where f is an arbitrary function of its arguments and $\varepsilon = (a_0, a_{11}, a_{12}, \dots, a_{33})$ is the group parameter such that the 3×3 -matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

has the property that $\det(A) = 1$. In particular, the equation is invariant under the special subgroup of the of the centro-affine group:

$$T_{\varepsilon_1} : \begin{cases} \tilde{x} = a_{11}x + a_{12}y + a_{13}z \\ \tilde{y} = a_{21}x + a_{22}y + a_{23}z \\ \tilde{z} = a_{31}x + a_{32}y + a_{33}z, \end{cases}$$

where $\varepsilon_1 = (a_{11}, a_{12}, \dots, a_{33})$.

3.2 Generalized equivalence transformations and equivalence transformations

The *equivalence transformations* that are analyzed in the case of differential equations with arbitrary functions have been introduced by [12] and generalized by [10]. For instance, the system (7) may be considered as belonging to a family \mathcal{T} of ODE systems – each function ϕ is related to a system of such form. Two systems of the family \mathcal{T} may have the same differential structure – here the differential structure is considered with respect to the derivatives of the dependent variables u – but differ from one another through the functions, say ϕ_1 and ϕ_2 . The two DE systems with the same differential structure that correspond to ϕ_1 and ϕ_2 are called *equivalent*. In this context, the system associated with ϕ_1 , for instance, may be named a *representative* of its corresponding equivalence class. By definition, an *equivalence transformation* related to the family \mathcal{T} (7) is a non-degenerate change of the independent variables t and the dependent variables u with the property that it leaves \mathcal{T} invariant. It maps a system of \mathcal{T} into another system belonging to \mathcal{T} . An equivalence transformation is given in its *finite form*, i.e., $t = P(\tilde{t}, \tilde{u})$ and $u = Q(\tilde{t}, \tilde{u})$ or through its related Lie group of transformations. The equivalence relation defined on \mathcal{T} divides this family into disjoint classes of equivalent systems. The systems of the family \mathcal{T} are permuted among themselves via distinct equivalence transformations [8]. The *equivalence group* consists in all equivalence transformations of this system.

To determine the equivalence transformation related to a family of differential equations, Ovsyannikov introduced a method based on the criterion for infinitesimal invariance which consists in seeking a Lie group of transformations of the form

$$\tilde{x} = F(x, u; \varepsilon), \quad \tilde{u} = G(x, u; \varepsilon), \quad \tilde{\phi} = H(x, u, \phi; \varepsilon),$$

where ε is the group parameter, that acts on the space of the independent and dependent variables of the system and the set of the arbitrary functions ϕ with the property that it leaves the differential structure of (7) invariant except for its arbitrary function. [10] generalized the notion of equivalence transformations by requesting that the above one-parameter Lie group of transformations has the form

$$\tilde{x} = f(x, u, \phi; \varepsilon), \quad \tilde{u} = g(x, u, \phi; \varepsilon), \quad \tilde{\Phi} = h(x, u, \phi; \varepsilon),$$

where ε is the group parameter, and, therefore, both \tilde{x} and \tilde{u} are sought as functions of ϕ . Throughout this paper, we will refer to the equivalence transformations in sense of Meleshko as *generalized equivalence transformations*. Therefore, a generalized equivalence transformation is a non-degenerate change of the dependent variables, independent variables, and arbitrary functions of the above form which transforms a differential equation system of a given class to a system of equations of the same class. Let us discuss the method of finding the generalized equivalence transformations related to a class of differential equation systems.

To determine the generalized equivalence transformations for the Tzitzeica curve equation, we apply the new method presented in [4] that has been implemented as the MAPLE routine GENDEFGET. The output of the routine GENDEFGET is the set of the coefficients of the generalized equivalence transformation operator or the coefficients of the equivalence transformation operator, depending on the user's request. The GENDEFGET procedure requires five input arguments. The first four arguments must be entered as lists: the DE system, the dependent variables, the arbitrary functions, and the independent variables. The fifth argument is 1 in the case of the generalized equivalence transformations and, respectively, 2 for equivalence transformations. If the input for the fifth argument is neither 1 nor 2, the user is prompted to input one of the above options.

In the case of the Tzitzeica curve equation (4), we obtain

Proposition 3. *The generalized equivalence transformations for the Tzitzeica curve equation are generated by the following vector fields*

$$\begin{aligned} X_1 &= y\partial_x, & X_2 &= x\partial_y, & X_3 &= y\partial_y - z\partial_z, \\ X_4 &= x\partial_x - z\partial_z, & X_5 &= x\partial_z, & X_6 &= y\partial_z, & X_7 &= m(t, x, y)\partial_t, \end{aligned} \quad (8)$$

where $m = m(t, x, y)$ is an arbitrary function of its arguments.

Proposition 4. *The equivalence transformations for the Tzitzeica curve equation are generated by the vector fields*

$$\begin{aligned} X_1 &= y\partial_x, & X_2 &= x\partial_y, & X_3 &= y\partial_y - z\partial_z, \\ X_4 &= x\partial_x - z\partial_z, & X_8 &= n(t)\partial_t, \end{aligned} \quad (9)$$

where $n = n(t)$ is an arbitrary function.

4 Conclusion

In this paper we discuss the extended classical symmetries, generalized equivalence transformations, and equivalence transformations related to one of the ODEs that arise from differential geometry, namely, the Tzitzeica curve equation. It is intriguing to see that the extended classical symmetries are, in fact, the centro-affine transformations. Therefore, the Tzitzeica curves are invariant under the centro-affine transformations because the nonlinear ODE that defines them has its related group of extended classical symmetries given by the group of centro-affine transformations. In a future work, it will be interesting to use these classes of symmetry reductions to determine new closed-form solutions. Notice that the Tzitzeica curve equation is a nonlinear ODE that may be regarded as an equation in one of the unknown functions, say z , and x and y arbitrary functions or as an equation in three unknown functions x , y , and z . This implies that there are many ways in which one can analyze specific symmetry reductions for the Tzitzeica curve equation.

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