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Abstract

The aim of this paper is to study a particular reduction case of the nonlinear ordinary differential equation that defines a Tzitzeica curve. It is shown that the Tzitzeica curve equation can be reduced to an auxiliary third order linear homogeneous ordinary differential equation with constant coefficients for the defining functions of the curve and a linear equation for the equation's constant. Consequently, it can be proven that any three linearly independent solutions of the auxiliary ordinary differential equation define a Tzitzeica curve.

Key-words: Nonlinear ordinary differential equation, Tzitzeica curves
Subject Classification: 34A05, 34A30, 34A34, 53A04, 53A15

1 Introduction

A *Tzitzeica curve* is a spatial curve for which the ratio of its torsion τ and the square of the distance d from the origin to the osculating plane at any arbitrary point of the curve is constant, i.e.,

$$\frac{\tau}{d^2} = \alpha, \tag{1}$$

where $\alpha \neq 0$ is a real constant. The curves with the property (1) have been introduced in 1911 by the Romanian mathematician Gheorghe Tzitzeica [5]

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as a consequence of his work on a particular class of surfaces that also carries his name today. A *Tzitzeica surface* has the property that the ratio of the surface's Gaussian curvature and the fourth power of the distance from the origin to the tangent plane at any arbitrary point of the surface is constant [6]. It may be shown that the asymptotic lines on any Tzitzeica surface with negative Gaussian curvature are Tzitzeica curves. One of the most intriguing properties of the Tzitzeica's geometric objects is their invariance under centro-affine transformations.

Although the Tzitzeica curves have occurred occasionally in the mathematics literature, their related ordinary differential equation (ODE) given by (4) has not been studied in detail so far. Moreover, there are known only a few examples of Tzitzeica curves whose defining functions are given explicitly (see, for instance, [1] and [3]). Therefore, our work is motivated by the aim of finding new closed-form solutions of the Tzitzeica curve equation. We explore an interesting case when the defining functions of a Tzitzeica curve are linearly independent and satisfy the third order linear homogeneous ODE with constant coefficients (7). We show that, in this situation, (7) reduces to (11) and, in addition, the Tzitzeica curve equation turns into (13) which may be seen as a linear equation for the constant α. As a consequence, we obtain new Tzitzeica curves (see Cases 1,2, and 3 along with the examples in Section 2). To the best of our knowledge, these representations of the Tzitzeica curves do not appear in literature.

The structure of the paper is the following. The Tzitzeica curve equation is derived in Section 2. In Section 3, a specific reduction case is analyzed and the resulting solutions are presented. Section 4 is reserved for conclusions.

2 The Tzitzeica curve equation

Let

$$\mathbf{r}(t) = (x(t), y(t), z(t)) , \tag{2}$$

be a regular space curve given parametrically with nonzero curvature k, where $t \in I \subset \mathbf{R}$. The torsion of the curve is given by

$$\tau(t) = \frac{(\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t))}{||\mathbf{r}'(t) \times \mathbf{r}''(t)||^2},$$

where the primes denote the derivatives with respect to t, the vector $\mathbf{r}' \times \mathbf{r}''$ is the cross product of the tangent vector $\mathbf{r}'$ and the acceleration vector $\mathbf{r}''$,

$||\mathbf{r}'(t) \times \mathbf{r}''(t)||$ is the magnitude of $\mathbf{r}' \times \mathbf{r}''$, and

$$(\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t)) = \begin{vmatrix} x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \\ x'''(t) & y'''(t) & z'''(t) \end{vmatrix}$$

is the mixed product (or the scalar triple product) of vectors $\mathbf{r}'$, $\mathbf{r}''$, and $\mathbf{r}'''$ (see, for instance, [4], page 48). Assume that (2) has a nonzero torsion. On the other hand, the osculating plane at an arbitrary point of the curve is given in the determinant form as

$$\begin{vmatrix} x - x(t) & y - y(t) & z - z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix} = 0.$$

The osculating plane is generated by the unit tangent vector $\mathbf{T}(t)$ and the unit normal vector $\mathbf{N}(t)$ at each point of the curve or, equivalently, by the tangent vector $\mathbf{r}'(t)$ and the acceleration vector $\mathbf{r}''(t)$. It may be shown that the distance from the origin to the osculating plane of the curve is

$$d^2 = \frac{1}{||\mathbf{r}'(t) \times \mathbf{r}''(t)||^2} \begin{vmatrix} x(t) & y(t) & z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix}^2 .$$

After substituting the torsion τ and the expression d^2 into (1), we obtain the following equation

$$\begin{vmatrix} x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \\ x'''(t) & y'''(t) & z'''(t) \end{vmatrix} = \alpha \begin{vmatrix} x(t) & y(t) & z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix}^2 \tag{3}$$

which may also be written as

$$az''' - a'z'' + bz' = \alpha \left(cz'' - c'z' + az\right)^2 , \tag{4}$$

where

$$a = x'y'' - x''y', \quad b = x''y''' - x'''y'', \quad \text{and} \quad c = xy' - x'y$$

are functions of the curve parameter t.

Proposition. *A space curve (2) is a Tzitzeica curve if and only if its defining functions x, y, and z satisfy the nonlinear ODE (4).*

Definition. The nonlinear ODE (4) is called the *Tzitzeica curve equation.*

3 A special case of reduction for the Tzitzeica curve equation

Note that the equation (4) may be written in terms of Wronskians as follows

$$W(x', y', z')(t) = \alpha \left[W(x, y, z)(t)\right]^2, \tag{5}$$

where

$$W(x, y, z)(t) = \begin{vmatrix} x(t) & y(t) & z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix}.$$

This means that the Wronskian $W(x', y', z')(t)$ and the square of the Wronskian $W(x, y, z)(t)$ are directly proportional. On the other hand, since a determinant is invariant under a cyclic permutation of its rows, the left-hand side of the equation (4) may be reformulated as below

$$\begin{vmatrix} x'''(t) & y'''(t) & z'''(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix} = \alpha \begin{vmatrix} x(t) & y(t) & z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix}^2. \tag{6}$$

Observe that if the functions x, y, and z satisfy the equations

$$\begin{aligned} x''' + \beta x'' + \gamma x' + \delta x = 0, \\ y''' + \beta y'' + \gamma y' + \delta y = 0, \\ z''' + \beta z'' + \gamma z' + \delta z = 0, \end{aligned} \tag{7}$$

where β, γ, and $\delta \neq 0$ are real numbers, the two determinants in (6) are equal (here we consider $\delta \neq 0$ because the curve's torsion should be nonzero). The above equations (7) mean that the functions x, y, and z are solutions of the third order linear homogeneous ODE with constant coefficients

$$u''' + \beta u'' + \gamma u' + \delta u = 0, \tag{8}$$

in the unknown function $u = u(t)$, where β, γ, and δ are real numbers with $\delta \neq 0$. Moreover, the solutions x, y, and z are chosen such that they are linearly independent (this restriction is related to the condition that the curve's torsion is nonzero). Next, the substitution of (7) into (6) implies

$$-\delta \begin{vmatrix} x(t) & y(t) & z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix} = \alpha \begin{vmatrix} x(t) & y(t) & z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix}^2,$$

or, equivalently,

$$-\delta W(t) = \alpha[W(t)]^2,$$

for any $t \in I$, where we denote $W(t) = W(x, y, z)(t)$. Since $W(t)$ is nonzero on the interval I (the functions x, y, and z are assumed to be linearly independent), the solution of the above equation is

$$W(t) = -\frac{\delta}{\alpha}. \tag{9}$$

Hence, the Tzitzeica curve equation has been reduced to the ODEs (8) and (9). By applying the Abel's differential equation identity that relates the derivative of the Wronskian of three homogeneous solutions of a third-order linear ODE and one of the coefficients of the original ODE (see, for example, [2], page 225), we get

$$W'(t) = -\beta W(t). \tag{10}$$

After replacing (9) into (10) and taking into account that $W(t)$ is nonzero, we obtain $\beta = 0$. Thus, the linear ODE (8) turns into

$$u''' + \gamma u' + \delta u = 0. \tag{11}$$

In what follows, we solve the differential equation (11) and discuss its solutions in each case. We start with the characteristic equation related to (11) which is given by

$$v^3 + \gamma v + \delta = 0. \tag{12}$$

Notice that v_1, v_2, and v_3 are nonzero because $\delta \neq 0$ and the sum of the roots is zero: $v_1 + v_2 + v_3 = 0$ (the coefficient of v^2 is zero). In the same time, the equation (9) to which the Tzitzeica curve equation is reduced, may be seen as a linear equation for α, i.e.,

$$\alpha = -\frac{\delta}{W}. \tag{13}$$

The following cases may occur:

Case 1. The equation (12) has three real nonzero simple roots whose sum is zero: $v_1 \neq 0$, $v_2 \neq 0, v_1$, and $v_3 = -v_1 - v_2$. Since $v_3 \neq v_1$ and $v_3 \neq v_2$, it follows that $v_2 \neq -2v_1$ and $v_2 \neq -v_1/2$. In this case, the general solution of the ODE (11) is

$$u(t) = C_1 \exp(v_1 t) + C_2 \exp(v_2 t) + C_3 \exp[-(v_1 + v_2)t], \quad t \in \mathbf{R}, \tag{14}$$

where C_1, C_2, and C_3 are real constants. Since each centro-affine transformation of a Tzitzeica curve is also a Tzitzeica curve, without loss of generality, it is enough to consider x, y, and z as the simplest fundamental set of solutions of the differential equation (11). Thus, the following functions

$$x(t) = \exp(v_1 t), \quad y(t) = \exp(v_2 t), \quad z(t) = \exp[-(v_1 + v_2)t] \tag{15}$$

define a Tzitzeica curve. In this case, the Wronskian of the functions x, y, and z is a nonzero constant given by

$$W(t) = (v_2 - v_1)(2v_1 + v_2)(v_1 + 2v_2).$$

From the equation (13), we find that

$$\alpha = \frac{v_1 v_2 (v_1 + v_2)}{(v_2 - v_1)(2v_1 + v_2)(v_1 + 2v_2)}.$$

Example 1. Consider the ODE $u''' - 7u' + 6u = 0$ whose characteristic equation has the roots $v_1 = 1$, $v_2 = 2$, and $v_3 = -3$. After replacing them into (15), we obtain the following Tzitzeica curve

$$x(t) = \exp(t), \quad y(t) = \exp(2t), \quad z(t) = \exp(-3t) \tag{16}$$

whose graph is given in Figure 1.

Case 2. Assume that equation (12) has a double real nonzero root $v_1 = v_2$ and a simple real nonzero root v_3. Since the sum of the roots is zero, we have $v_3 = -2v_1$. It follows that the general solution of the ODE (11) is given by

$$u(t) = C_1 \exp(v_1 t) + C_2 t \exp(v_1 t) + C_3 \exp(-2v_1 t), \tag{17}$$

where C_1, C_2, and C_3 are real constants. Therefore,

$$x(t) = \exp(v_1 t), \quad y(t) = t \exp(v_1 t), \quad z(t) = \exp(-2v_1 t) \tag{18}$$

define a Tzitzeica curve. The Wronskian of the functions in (18) is $W(t) = 9v_1^2 \neq 0$. The resulting equation (13) implies that the equation's constant may be chosen as $\alpha = 2v_1/9$.

Example 2. Consider the differential equation $u''' - 3u' + 2u = 0$. In this case, the characteristic equation has the roots $v_1 = v_2 = 1$ and $v_3 = -2$. By replacing these numbers in (18), we obtain the following Tzitzeica curve

$$x(t) = \exp(t), \quad y(t) = t \exp(t), \quad z(t) = \exp(-2t) \tag{19}$$

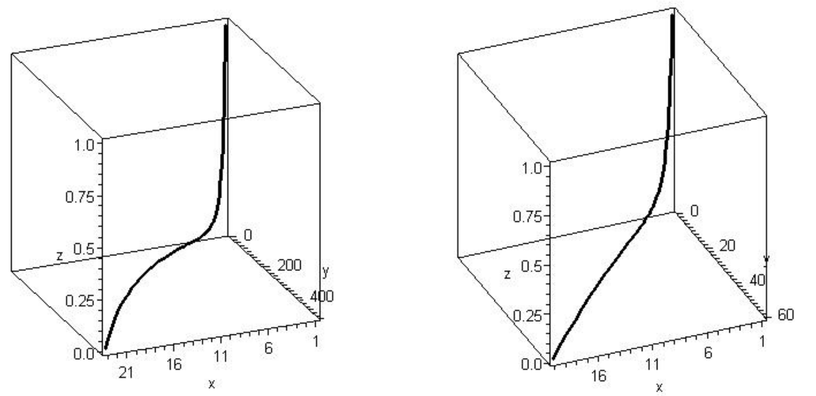

Figure 1: *The Tzitzeica curves defined by (16) and, respectively, (19).*

whose graphical representation is given in Figure 1.

Case 3. The equation (12) has two nonzero complex conjugate roots $v_{1,2} = m \pm in$, where $m \neq 0$ and n are real numbers and a simple real nonzero root v_3. Since the sum of the roots is zero, we have that $v_3 = -2m$. The general solution of the equation (11) is

$$u(t) = C_1 \exp(mt)\cos(nt) + C_2 \exp(mt)\sin(nt) + C_1 \exp(-2mt), \qquad (20)$$

where C_1, C_2, and C_3 are real constants. Therefore, the functions

$$x(t) = \exp(mt)\cos(nt),\ y(t) = \exp(mt)\sin(nt),\ z(t) = \exp(-2mt) \qquad (21)$$

define a Tzitzeica curve. The Wronskian of the above functions is $W(t) = n(9m^2 + n^2)$. Next, we use the remaining equation (13) to determine equation's constant, that is,

$$\alpha = \frac{2m(m^2+n^2)}{n(9m^2+n^2)}.$$

Example 3. Consider the ODE $u''' - u = 0$. The roots of the characteristic equation are $v_1 = 1$ and $v_{2,3} = -1/2 \pm i\sqrt{3}/2$. The relations (21) imply

$$x(t) = \exp\left(-\frac{t}{2}\right)\cos\left(\frac{\sqrt{3}}{2}t\right),\ y(t) = \exp\left(-\frac{t}{2}\right)\sin\left(\frac{\sqrt{3}}{2}t\right),\ z(t) = \exp(t), \qquad (22)$$

where $t \in \mathbf{R}$. The graph of this Tzitzeica curve is given in Figure 2.

Notice that the case when the equation (12) has a triple real root $v_1 = v_2 = v_3$ does not lead to any Tzitzeica curve. Indeed, since the sum of the roots is zero, one obtains $v_1 = 0$ and this is in contradiction with $\delta \neq 0$.

We have shown the following result:

Theorem. *Any linearly independent solutions of the third order linear homogeneous ODE with constant coefficients (11) define a Tzitzeica curve equation. The corresponding families of Tzitzeica curves are given by (15), (18), and (21).*

4 Conclusion

The Tzitzeica curve equation (4) may be regarded as a nonlinear ODE in one of the unknown functions, say z, with x and y arbitrary functions or as a nonlinear ODE in all three unknown functions x, y, and z. In both cases, it is difficult to find closed-form solutions, even with computer assistance. In this paper, we have discussed a particular case that follows from the observation that the Tzitzeica curve equation can be expressed in terms of two Wronskians (5). This key remark led us to the assumption that the defining functions of a Tzitzeica curve satisfy a third order linear homogeneous ODE. For simplicity, we have discussed the case when x, y, and z satisfy a linear ODE with constant coefficients but it would be interesting in a future study to analyze also the case when the auxiliary linear ODE has variable coefficients. In Section 3, we have shown that if the defining functions of a Tzitzeica curve are three linearly independent solutions of a linear homogeneous ODE with constant coefficients (8), then in this equation the coefficient of u'' must be zero. Moreover, the constant α may be determined from the linear equation (13) that represents the corresponding reduced form of the Tzitzeica curve equation. Here, we should specify that the resulting linear equation (13) may also be regarded as a restriction on the roots v_1, v_2, and v_3. For instance, if α is fixed and given, we can find v_1 in terms of v_2 and α (in Case 1), $v_1 = 9\alpha/2$ (in Case 2), and m in terms of n and α (in Case 3). We also provide the new families of closed-form solutions that are obtained in each situation. Since any centro-affine transformation of a Tzitzeica curve is also a Tzitzeica curve, one can use this property to construct new Tzitzeica curves from the known ones. That is, given a Tzitzeica curve and a centro-affine

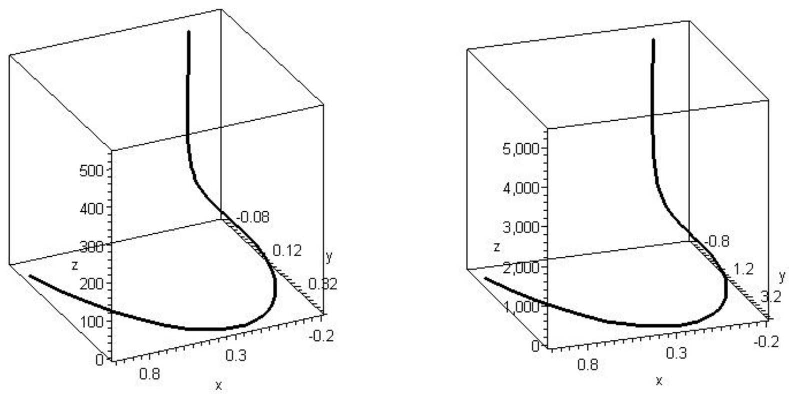

Figure 2: *The Tzitzeica curve (22) and its deformation by the affine transformation $\tilde{x} = x$, $\tilde{y} = 10y$, and $\tilde{z} = 10z$.*

transformation $[x, y, z]^T \to [\tilde{x}, \tilde{y}, \tilde{z}]^T = A[x, y, z]^T$, where A is a 3×3 -matrix with $\det(A) \neq 0$, the resulting curve defined by $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$ is also a Tzitzeica curve. For instance, in Figure 2, a simple example of a deformation of the Tzitzeica curve (22) under the centro-affine transformation $\tilde{x} = x$, $\tilde{y} = 10y$, and $\tilde{z} = 10z$ is presented. The original curve corresponds to $\alpha = 2\sqrt{3}/9$ while the new curve has $\alpha = \sqrt{3}/450$. It may be shown that under a central equi-affine transformation, i.e., $[x, y, z]^T \to [\tilde{x}, \tilde{y}, \tilde{z}]^T = A[x, y, z]^T$, with $\det(A) = 1$, the original and the transformed curves correspond to the same constant α.

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