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Xin Tang's paper on the Hopf algebra, please see .pdf

Xin Tang
Fayetteville State University, xtang@uncfsu.edu

(HOPF) ALGEBRA AUTOMORPHISMS OF THE HOPF ALGEBRA $\check{U}_{r,s}^{\geq 0}(\mathfrak{s}l_3)$

XIN TANG

Abstract. In this paper, we completely determine the group of algebra automorphisms for the two-parameter Hopf algebra $\check{U}_{r,s}^{\geq 0}(\mathfrak{s}l_3)$. As a result, the group of Hopf algebra automorphisms is determined for $\check{U}_{r,s}^{\geq 0}(\mathfrak{s}l_3)$ as well. We further characterize all the derivations of the subalgebra $U_{r,s}^{+}(\mathfrak{s}l_3)$, and calculate its first degree Hochschild cohomology group.

Introduction

Motivated by the study of down-up algebras [2], a two-parameter quantized enveloping algebra (quantum group) $U_{r,s}(\mathfrak{s}l_n)$ has recently been investigated by Benkart and Witherspoon in the references [3, 4]. The two-parameter quantized enveloping algebra $U_{r,s}(\mathfrak{s}l_n)$ remains as a close analogue of the one-parameter quantized enveloping algebra $U_q(\mathfrak{s}l_n)$ associated to the finite dimensional complex simple special linear Lie algebra $\mathfrak{s}l_n$. As a matter of fact, the two-parameter quantized enveloping algebra $U_{r,s}(\mathfrak{s}l_n)$ shares many similar properties with its one-parameter analogue.

For instance, the two-parameter quantized enveloping algebra $U_{r,s}(\mathfrak{s}l_n)$ is also a Hopf algebras and it admits a triangular decomposition. Besides having a similar representation theory to that of the one-parameter quantum group $U_q(\mathfrak{s}l_n)$, the Hopf algebra $U_{r,s}(\mathfrak{s}l_n)$ can be realized as the Drinfeld double of its Hopf sub-algebras $U_{r,s}^{\geq 0}(\mathfrak{s}l_n)$ and $U_{r,s}^{\leq 0}(\mathfrak{s}l_n)$. However, the algebra $U_{r,s}(\mathfrak{s}l_n)$ does have some different ring-theoretic features in that $U_{r,s}(\mathfrak{s}l_n)$ is more rigid and possesses less symmetries. In addition, the center of the algebra $U_{r,s}(\mathfrak{s}l_n)$ shows a different picture as well [1].

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To better understand the structure and properties of the algebra $U_{r,s}(\mathfrak{sl}_n)$, one naturally starts with the investigations of its subalgebra $U^+_{r,s}(\mathfrak{sl}_n)$ and Hopf subalgebra $U^{\geq 0}_{r,s}(\mathfrak{sl}_n)$. In this paper, we will study the algebra $U^+_{r,s}(\mathfrak{sl}_3)$ in terms of its derivations, and its augmented Hopf algebra $\check{U}^{\geq 0}_{r,s}(\mathfrak{sl}_3)$ in terms of its algebra automorphisms and Hopf algebra automorphisms. We shall determine all the algebra automorphisms and Hopf algebra automorphisms of $\check{U}^{\geq 0}_{r,s}(\mathfrak{sl}_3)$. In addition, we will characterize all the derivations of $U^+_{r,s}(\mathfrak{sl}_3)$. As a result, we will compute the first Hochschild cohomology group of the algebra $U^+_{r,s}(\mathfrak{sl}_3)$.

Now let me briefly mention the methods which we shall follow. In order to determine the algebra and Hopf algebra automorphisms of $\check{U}^{\geq 0}_{r,s}(\mathfrak{sl}_3)$, we shall closely follow the approach used in [5]. In order to characterize the derivations of $U^+_{r,s}(\mathfrak{sl}_3)$, we shall embed the algebra $U^+_{r,s}(\mathfrak{sl}_3)$ into a quantum torus, whose derivations had been explicitly described in [7]. We would like to point out that this embedding will allow us to extend the derivations of $U^+_{r,s}(\mathfrak{sl}_3)$ to the derivations of the associated quantum torus. Therefore, via this embedding, we shall be able to pull the information on derivations back to the algebra $U^+_{r,s}(\mathfrak{sl}_3)$. Based on a result on the derivations of quantum torus established in [7], we will be able to determine all the derivations of the algebra $U^+_{r,s}(\mathfrak{sl}_3)$ modulo its inner derivations. As an immediate application, we show that the first Hochschild cohomology group $HH^1(U^+_{r,s}(\mathfrak{sl}_3))$ of $U^+_{r,s}(\mathfrak{sl}_3)$ is a $2-$dimensional vector space over the base field $\mathbb{C}$.

The paper is organized as follows. In Section 1, we recall some basic definitions and properties on the two-parameter quantized enveloping algebras $U^+_{r,s}(\mathfrak{sl}_3)$ and its augmented Hopf algebra $\check{U}^{\geq 0}_{r,s}(\mathfrak{sl}_3)$. In Section 2, we determine the algebra automorphism group and Hopf algebra automorphism group of $\check{U}^{\geq 0}_{r,s}(\mathfrak{sl}_3)$. In Section 3, we characterize the derivations of $U^+_{r,s}(\mathfrak{sl}_3)$, and compute the first Hochschild cohomology group $HH^1(U^+_{r,s}(\mathfrak{sl}_3))$.

1. Definitions and Basic Properties of $U^+_{r,s}(\mathfrak{sl}_3)$ and $\check{U}^{\geq 0}_{r,s}(\mathfrak{sl}_3)$

1.1. **Definition and basic properties of $U^+_{r,s}(\mathfrak{sl}_3)$.** Motivated by the study of down-up algebras, a two-parameter quantized enveloping algebra $U_{r,s}(\mathfrak{sl}_n)$ associated to the finite dimensional simple complex Lie algebra $\mathfrak{sl}_n$ has been recently studied by Benkart and Witherspoon in [3] and the references therein. For the purpose of this paper, we shall only recall the definitions of the algebras $U^+_{r,s}(\mathfrak{sl}_n)$ and $U^{\geq 0}_{r,s}(\mathfrak{sl}_n)$, which are indeed subalgebras of $U_{r,s}(\mathfrak{sl}_n)$. One easily sees that the algebra

$U_{r,s}^{+}(\mathfrak{sl}_n)$ can be regarded as a two-parameter quantized enveloping algebra of a maximal nilpotent Lie subalgebra of the Lie algebra $\mathfrak{sl}_n$.

Let $C = (a_{ij})$ denote the Cartan matrix associated to the Lie algebra $\mathfrak{sl}_n$. Let us define the following notation:

$$\begin{aligned} < i, j >&= a_{ij} \,\text{for}\, i < j; \\ < i, i >&= 1 \,\text{for}\, i = 1, \cdots, n-1; \\ < i, j >&= 0 \,\text{for}\, i > j. \end{aligned}$$

Suppose that $r, s \in \mathbb{C}$ such that $r^m s^n = 1$ implies $m = n = 0$. We recall the following definition:

Definition 1.1. The two-parameter quantized enveloping algebra $U_{r,s}^{\geq 0}(\mathfrak{sl}_n)$ is defined to be the $\mathbb{C}-$algebra generated by E_i, W_i subject to the following relations:

$$\begin{aligned} W_i^{\pm 1} W_j^{\pm 1} &= W_j^{\pm 1} W_i^{\pm 1}; \\ W_i^{\pm 1} W_i^{\mp 1} &= 1; \\ W_i E_j &= r^{<j,i>} s^{-<i,j>} E_j W_i; \\ E_i^2 E_{i+1} - (r+s) E_i E_{i+1} E_i + rs E_{i+1} E_i^2 &= 0; \\ E_{i+1}^2 E_i - (r^{-1} + s^{-1}) E_{i+1} E_i E_{i+1} + r^{-1} s^{-1} E_i E_{i+1}^2 &= 0. \end{aligned}$$

And the algebra $U_{r,s}^{+}(\mathfrak{sl}_n)$ is defined to be the subalgebra of $U_{r,s}^{\geq 0}(\mathfrak{sl}_n)$ generated by E_i.

From [3], we know that the two-parameter quantized enveloping algebra $U_{r,s}^{\geq 0}(\mathfrak{sl}_n)$ has a Hopf algebra structure, which is defined by the following coproduct, counit and antipode:

$$\begin{aligned} \Delta(W_i^{\pm 1}) &= W_i^{\pm 1} \otimes W_i^{\pm 1}; \\ \Delta(E_i) &= E_i \otimes 1 + W_i \otimes E_i; \\ \epsilon(W_i^{\pm 1}) &= 1; \\ \epsilon(E_i) &= 0; \\ S(W_i^{\pm 1}) &= W_i^{\mp 1}; \\ S(E_i) &= -W_i^{-1} E_i. \end{aligned}$$

When $n = 3$, one has the corresponding two-parameter quantized enveloping algebra $U_{r,s}^{+}(\mathfrak{sl}_3)$, which will be of one of the major objects in this paper. In particular, we recall the following definition:

Definition 1.2. The algebra $U^+_{r,s}(\mathfrak{sl}_3)$ is defined to be the $\mathbb{C}-$algebra generated by the generators E_1, E_2 subject to the following relations:

$$E_1^2E_2 - (r+s)E_1E_2E_1 + rsE_2E_1^2 = 0;$$
$$E_1E_2^2 - (r+s)E_1E_2E_1 + rsE_2^2E_1 = 0.$$

Naturally, one may think of the algebra $U^+_{r,s}(\mathfrak{sl}_3)$ as a two-parameter quantum Heisenberg algebra. Indeed, the algebra $U^+_{r,s}(\mathfrak{sl}_3)$ shares many similar properties as the algebra $U^+_q(\mathfrak{sl}_3)$, which has been traditionally called the quantum Heisenberg algebra.

In addition, we recall the definition of the following Hopf subalgebra algebra of $U_{r,s}(\mathfrak{sl}_3)$:

Definition 1.3. The Hopf algebra $U^{\geq 0}_{r,s}(\mathfrak{sl}_3)$ is defined to be the $\mathbb{C}-$algebra generated by E_1, E_2, W_1, W_2 subject to the following relations:

$$\begin{aligned}
W_1W_1^{-1} = 1 = W_2W_2^{-1};\\
W_1W_2 = W_2W_1;\\
W_1E_1 = rs^{-1}E_1W_1;\\
W_1E_2 = sE_2W_1;\\
W_2E_1 = r^{-1}E_1W_2;\\
W_2E_2 = rs^{-1}E_2W_2;\\
E_1^2E_2 - (r+s)E_1E_2E_1 + rsE_2E_1^2 = 0;\\
E_1E_2^2 - (r+s)E_2E_1E_2 + rsE_2^2E_1 = 0.
\end{aligned}$$

However, we should mention that we will not study the Hopf algebra $U^{\geq 0}_{r,s}(\mathfrak{sl}_3)$ in this paper. Instead, we will study its augmented version $\check{U}^{\geq 0}_{r,s}(\mathfrak{sl}_3)$, which we shall define in the next subsection.

Before we introduce the Hopf algebra $\check{U}^{\geq 0}_{r,s}(\mathfrak{sl}_3)$, let us mention some basic properties of the algebra $U^+_{r,s}(\mathfrak{sl}_3)$ in the rest of this subsection. It is easy to see that the two-parameter quantized enveloping algebra $U^+_{r,s}(\mathfrak{sl}_n)$ can also be presented as an iterated skew polynomial ring and a PBW-basis can be constructed for $U_{r,s}(\mathfrak{sl}_n)$ as well. For conveniences, we shall only recall the skew polynomial presentation for the algebra $U^+_{r,s}(\mathfrak{sl}_3)$ here. For the general construction, we refer the reader to references [1, 8].

First of all, Let us fix some notation by setting the following new variables:

$$E_1 = E_1, \quad E_2 = E_2, \quad E_3 = E_1E_2 - sE_2E_1.$$

Then it is easy to see that we have the following relations between these new variables:

$$E_1E_3 = rE_3E_1, \quad E_2E_3 = r^{-1}E_3E_2.$$

Now let us further define some algebra automorphisms τ_2, τ_3 and some derivations δ_2, δ_3 as follows:

$$\begin{aligned}
\tau_2(E_1) &= r^{-1}E_1;\\
\delta_2(E_1) &= 0;\\
\tau_3(E_1) &= s^{-1}E_1;\\
\tau_3(E_3) &= r^{-1}E_3;\\
\delta_3(E_1) &= -s^{-1}E_3;\\
\delta_3(E_3) &= 0.
\end{aligned}$$

□

Then it is easy to see that we have the following result

Theorem 1.1. *The algebra $U^+_{r,s}(\mathfrak{s}l_3)$ can be presented as an iterated skew polynomial ring. In particular, we have the following result*

$$U^+_{r,s}(\mathfrak{s}l_3) \cong \mathbb{C}[E_1][E_3, \tau_2, \delta_2][E_2, \tau_3, \delta_3].$$

□

Based on the previous theorem, we have an obvious corollary as follows:

Corollary 1.1. *The set $\{E_1^iE_3^jE_2^k | i, j, k \geq 0\}$ forms a PBW-basis of the algebra $U^+_{r,s}(\mathfrak{s}l_3)$. In particular, $U^+_{r,s}(\mathfrak{s}l_3)$ has a $GK-$dimension of 3.*

□

1.2. **The Augmented Hopf algebra $\check{U}^{\geq 0}_{r,s}(\mathfrak{s}l_3)$.** In this subsection, we shall introduce an augmented Hopf algebra $\check{U}^{\geq 0}_{r,s}(\mathfrak{s}l_3)$, which contains the algebra $U^+_{r,s}(\mathfrak{s}l_3)$ as a subalgebra and enlarges the Hopf algebra $U^{\geq 0}_{r,s}((sl_3)$.

First of all, we need to define the following new variables:

$$K_1 = W_1^{2/3}W_2^{1/3}, \quad K_2 = W_1^{1/3}W_2^{2/3}$$

Then we have the following definition of $\check{U}^{\geq 0}_{r,s}(\mathfrak{s}l_3)$.

Definition 1.4. The algebra $\check{U}_{r,s}^{\geq 0}(\mathfrak{sl}_3)$ is a $\mathbb{C}$-algebra generated by $E_1, E_2, K_1^{\pm 1}, K_2^{\pm 1}$ subject to the following relations:

$$\begin{aligned} K_1K_1^{-1} &= 1 = K_2K_2^{-1};\\ K_1K_2 &= K_2K_1;\\ K_1E_1 &= r^{1/3}s^{-2/3}E_1K_1;\\ K_1E_2 &= r^{1/3}s^{1/3}E_2K_1;\\ K_2E_1 &= r^{-1/3}s^{-1/3}E_1K_2;\\ K_2E_2 &= r^{2/3}s^{-1/3}E_2K_2;\\ E_1^2E_2-(r+s)E_1E_2E_1+rsE_2E_1^2 &= 0;\\ E_1E_2^2-(r+s)E_2E_1E_2+rsE_2^2E_1 &= 0. \end{aligned}$$

□

In order to introduce a Hopf algebra structure on $\check{U}_{r,s}^{\geq 0}(\mathfrak{sl}_3)$, let us further define some operators as follows:

$$\begin{aligned} \Delta(E_1) &= E_1\otimes 1 + K_1^2K_2^{-1}\otimes E_1;\\ \Delta(E_2) &= E_2\otimes 1 + K_1^{-1}K_2^{2}\otimes E_2;\\ \Delta(K_1) &= K_1\otimes K_1;\\ \Delta(K_2) &= K_2\otimes K_2;\\ S(E_1) &= -K_1^2K_2^{-1}E_1;\\ S(E_2) &= -K_1^{-1}K_2^2E_1;\\ S(K_1) &= K_1^{-1};\\ S(K_2) &= K_2^{-1};\\ \epsilon(E_1) &= \epsilon(E_2)=0;\\ \epsilon(K_1) &= \epsilon(K_2)=1. \end{aligned}$$

Then it is straightforward to verify the following result:

Proposition 1.1. *The algebra $\check{U}_{r,s}^{\geq 0}(\mathfrak{sl}_3)$ is a Hopf algebra with the coproduct, counit and antipode defined as above.*

□

Recall that we have $E_3 = E_1E_2 - sE_2E_1$, then it is easy to see that we have the following result

Theorem 1.2. *The algebra $\check{U}_{r,s}^{\geq 0}(sl_3)$ has a $\mathbb{C}-$basis*

$$\{K_1^mK_2^nE_1^iE_2^jE_3^k \mid m,n\in\mathbb{Z},\ i,j,k\in\mathbb{Z}_{\geq 0}\}.$$

□

In particular, one can see that all the invertible elements of $\check{U}_{r,s}^{\geq 0}(\mathfrak{sl}_3)$ are of the form $\lambda K_1^mK_2^n$ for some $\lambda\in\mathbb{C}^*$ and $m,n\in\mathbb{Z}$.

2. Algebra and Hopf algebra automorphisms of $\check{U}^{\geq 0}_{r,s}(\mathfrak{sl}_3)$

In this section, we will first determine the algebra automorphism group of the algebra $\check{U}^{\geq 0}_{r,s}(\mathfrak{sl}_3)$. As a result, we are able to determine the Hopf algebra automorphism group of $\check{U}^{\geq 0}_{r,s}(\mathfrak{sl}_3)$ as well. We will closely follow the approach used in [5].

2.1. **The algebra automorphism group of $\check{U}^{\geq 0}_{r,s}(\mathfrak{sl}_3)$.** Suppose that $\theta \in Aut_{\mathbb{C}}(\check{U}^{\geq 0}_{r,s}(\mathfrak{sl}_3))$ is an algebra automorphism of the algebra $\check{U}^{\geq 0}_{r,s}(\mathfrak{sl}_3)$. Note that the elements K_1, K_2 are invertible in $\check{U}^{\geq 0}_{r,s}(\mathfrak{sl}_3)$ and θ is an algebra automorphism, then the elements $\theta(K_1)$ and $\theta(K_2)$ are invertible too. Therefore, we have the following

$$\theta(K_1) = a_1 K_1^x K_2^y, \quad \theta(K_2) = a_2 K_1^z K_2^w$$

for some $a_1, a_2 \in \mathbb{C}^*$ and $x, y, z, w \in \mathbb{Z}$.

Let $M_\theta = (M_{ij})$ denote the corresponding $2 \times 2-$matrix associated to the algebra automorphism *theta*. Specifically, we will set $M_{11} = x, M_{12} = y, M_{21} = z$ and $M_{22} = w$. Since θ is an algebra automorphism, we know that the matrix M_θ is an invertible matrix with integer coefficients. In particular, we have the following

$$xw - yz = \pm 1.$$

Suppose that for $l = 1, 2$, we have

$$\theta(E_l) = \sum_{m_l, n_l, \beta_l^1, \beta_l^2, \beta_l^3} a_{m_l, n_l, \beta_l^1, \beta_l^2, \beta_l^3} K_1^{m_l} K_2^{n_l} E_1^{\beta_l^1} E_2^{\beta_l^2} E_3^{\beta_l^3}$$

where $a_{m_l, n_l, \beta_l^1, \beta_l^2, \beta_l^3} \in \mathbb{C}^*$ and $m_l, n_l \in \mathbb{Z}$ and $\beta_l^1, \beta_l^2, \beta_l^3 \in \mathbb{Z}_{\geq 0}$.

Then we have the following

Proposition 2.1. *Let $\theta \in Aut_{\mathbb{C}}(\check{U}^{\geq 0}_{r,s}(\mathfrak{sl}_3))$ be an algebra automorphism of $\check{U}^{\geq 0}_{r,s}(\mathfrak{sl}_3)$, then we have $M_\theta \in GL(2, \mathbb{Z}_{\geq 0})$.*

Proof: The proof of **Proposition 2.1** in [5] can essentially be adopted here word by word with the replacement of s by r^{-1}. For the reader's convenience, we will present the detailed proof here.

Since $K_1 E_1 = r^{1/3} s^{-2/3} E_1 K_1$ and $K_2 E_1 = r^{-1/3} s^{-1/3} E_1 K_2$, we have the following

$$\theta(K_1)\theta(E_1) = r^{1/3} s^{-2/3} \theta(E_1)\theta(K_1);$$
$$\theta(K_2)\theta(E_1) = r^{-1/3} s^{-1/3} \theta(E_1)\theta(K_2).$$

Via a detailed calculation, we further have the following

$$(\beta_1^1 + \beta_1^3)x + (\beta_1^2 + \beta_1^3)y = 1;$$
$$(\beta_1^1 + \beta_1^3)z + (\beta_1^2 + \beta_1^3)w = 0.$$

Similarly, we also have the following

$$(\beta_2^1+\beta_2^3)x+(\beta_2^2+\beta_2^3)y=0$$
$$(\beta_2^1+\beta_2^3)z+(\beta_2^2+\beta_2^3)w=1.$$

Now let us set a $2\times 2-$matrix $B=(b_{ij})$ with the following entries

$$b_{11}=(\beta_1^1+\beta_1^3);$$
$$b_{21}=(\beta_1^2+\beta_1^3);$$
$$b_{12}=(\beta_2^1+\beta_2^3);$$
$$b_{22}=(\beta_2^2+\beta_2^3).$$

Then we have the following system of equations

$$b_{11}x+b_{21}y=1;$$
$$b_{12}x+b_{22}y=0;$$
$$b_{11}z+b_{21}w=0;$$
$$b_{12}z+b_{22}w=1.$$

This implies that the product $M_\theta B$ of matrices M_θ and B is equal to the identity matrix. Therefore, we have $M_{\theta^{-1}}=B$, where $M_{\theta^{-1}}$ is the corresponding matrix associated to the algebra automorphism θ^{-1}. Note that the entries $b_{11},b_{12},b_{21},b_{22}$ are all nonnegative integers, thus we have $M_{\theta^{-1}}\in GL(2,\mathbb{Z}_{\geq 0})$.

Applying similar arguments to the algebra automorphism θ^{-1}, we can prove that $M_\theta\in GL(2,\mathbb{Z}_{\geq 0})$ as desired. □

To proceed, we now recall an important lemma (**Lemma 2.2** from [5]), which characterizes the matrix M_θ:

Lemma 2.1. *If M is a matrix in $GL(n,\mathbb{Z}_{\geq 0})$ such that its inverse matrix M^{-1} is also in $GL(n,\mathbb{Z}_{\geq 0})$, then we have $M=(\delta_{i\sigma(j)})_{i,j}$, where σ is an element of the symmetric group $\mathbb{S}_n$.*

□

Based the previous Proposition and Lemma, we immediately have the following result

Corollary 2.1. *Let $\theta\in Aut_{\mathbb{C}}(\check{U}_{r,s}^{\geq 0}(\mathfrak{s}l_3))$ be an algebra automorphism of $\check{U}_{r,s}^{\geq 0}(\mathfrak{s}l_3)$. Then for $l=1,2$, we have*

$$\theta(K_l)=a_lK_{\sigma(l)}$$

where $\sigma\in\mathbb{S}_2$ and $a_l\in\mathbb{C}^*$.

□

Furthermore, we can further prove the following result:

Proposition 2.2. *Let $\theta \in Aut_{\mathbb{C}}(\check{U}_{r,s}^{\geq 0}(\mathfrak{s}l_3))$ be an algebra automorphism of $\check{U}_{r,s}^{\geq 0}(\mathfrak{s}l_3)$. Then for $l = 1, 2$, we have*

$$\theta(E_l) = b_l K_1^{m_l} K_2^{n_l} E_{\sigma(l)}$$

where $b_l \in \mathbb{C}^*$ and $m_l, n_l \in \mathbb{Z}$.

Proof: Let $\theta \in Aut_{\mathbb{C}}(\check{U}_{r,s}^{\geq 0}(\mathfrak{s}l_3))$ be an algebra automorphism of $\check{U}_{r,s}^{\geq 0}(\mathfrak{s}l_3)$. To prove the proposition, there are two cases to consider:

Case 1: Suppose that $\theta(K_1) = a_1 K_1$ and $\theta(K_2) = a_2 K_2$, then we need to prove that

$$\theta(E_1) = b_1 K_1^{m_1} K_2^{n_1} E_1, \quad \theta(E_2) = b_2 K_1^{m_2} K_2^{n_2} E_2.$$

Since $K_1 E_1 = r^{1/3} s^{-2/3} E_1 K_1$ and $K_2 E_1 = r^{-1/3} s^{-1/3} E_1 K_2$, we have the following

$$\theta(K_1)\theta(E_1) = r^{1/3} s^{-2/3} \theta(E_1)\theta(K_1);$$
$$\theta(K_2)\theta(E_1) = r^{-1/3} s^{-1/3} \theta(K_2)\theta(K_1).$$

Thus we have the following

$$\begin{aligned} & a_1 K_1 \Big(\sum_{m_1, n_1, \beta_1^1, \beta_1^2, \beta_1^3} b_{m_1, n_1, \beta_1^1, \beta_1^2, \beta_1^3} K_1^{m_1} K_2^{n_1} E_1^{\beta_1^1} E_2^{\beta_1^2} E_3^{\beta_1^3} \Big) \\ = \; & a_1 r^{1/3} s^{-2/3} \Big(\sum_{m_1, n_1, \beta_1^1, \beta_1^2, \beta_1^3} b_{m_1, n_1, \beta_1^1, \beta_1^2, \beta_1^3} K_1^{m_1} K_2^{n_1} E_1^{\beta_1^1} E_2^{\beta_1^2} E_3^{\beta_1^3} \Big) K_1. \end{aligned}$$

We also have the following

$$\begin{aligned} & a_2 K_2 \Big(\sum_{m_1, n_1, \beta_1^1, \beta_1^2, \beta_1^3} b_{m_1, n_1, \beta_1^1, \beta_1^2, \beta_2^3} K_1^{m_1} K_2^{n_1} E_1^{\beta_1^1} E_2^{\beta_1^2} E_3^{\beta_1^3} \Big) \\ = \; & a_2 r^{-1/3} s^{-1/3} \Big(\sum_{m_1, n_1, \beta_1^1, \beta_1^2, \beta_1^3} b_{m_1, n_1, \beta_1^1, \beta_1^2, \beta_1^3} K_1^{m_1} K_2^{n_1} E_1^{\beta_1^1} E_2^{\beta_1^2} E_3^{\beta_1^3} \Big) K_2. \end{aligned}$$

Via detailed calculations and simplifications, we have the following

$$\beta_1^1 + \beta_1^3 = 1, \quad \beta_1^2 + \beta_1^3 = 0.$$

Based on the fact that β_i^j are nonnegative integers, we have that

$$\beta_1^1 = 1, \quad \beta_1^2 = 0 = \beta_1^3.$$

Similarly, we can also have the following

$$\beta_2^1 = \beta_2^3 = 0, \quad \beta_2^2 = 1.$$

Thus we have proved **Case 1**.

Case 2: Suppose that $\theta(K_1) = a_1 K_2$ and $\theta(K_2) = a_2 K_1$, we need to prove that $\theta(E_1) = b_1 K_1^{m_1} K_2^{n_1} E_2$ and $\theta(E_2) = b_2 K_1^{m_2} K_2^{n_2} E_1$. Since the proof goes the same as in **Case 1**, we will not repeat the detail here. $\square$

Furthermore, we can easily verify that E_1, E_2 can not be exchanged. In particular, we have the following result

Corollary 2.2. *Let $\theta \in Aut_{\mathbb{C}}(\check{U}_{r,s}^{\geq 0}(\mathfrak{s}l_3))$ be an algebra automorphism. Then for $l = 1, 2$, we have*

$$\theta(K_l) = a_l K_l,\ \theta(E_l) = b_l K_1^{m_l} K_2^{n_l} E_l$$

where $a_l, b_l \in \mathbb{C}^*$ and $m_l, n_l \in \mathbb{Z}$.

$\square$

Now we are going to prove one of the main results of this paper, which describes the algebra automorphism group of $\check{U}_{r,s}^{\geq 0}(\mathfrak{s}l_3)$:

Theorem 2.1. *Let $\theta \in Aut_{\mathbb{C}}(\check{U}_{r,s}^{\geq 0}(\mathfrak{s}l_3))$ be an algebra automorphism of $\check{U}_{r,s}^{\geq 0}(\mathfrak{s}l_3)$. Then for $l = 1, 2$, we have the following*

$$\theta(K_l) = a_l K_l, \quad \theta(E_1) = b_1 K_1^a K_2^b E_1, \quad \theta(E_2) = b_2 K_1^c K_2^d E_2$$

where $a_l, b_l \in \mathbb{C}^*$ and $a, b, c, d \in \mathbb{Z}$ such that $b = c, a + b + d = 0$.

Proof: Let θ be an algebra automorphism of $\check{U}_{r,s}^{\geq 0}(\mathfrak{s}l_3)$ and suppose that

$$\theta(E_1) = b_1 K_1^a K_2^b E_1, \quad \theta(E_2) = b_2 K_1^c K_2^d E_2.$$

Then we have the following

$$\begin{aligned}
(K_1^a K_2^b E_1)(K_1^a K_2^b E_1)(K_1^c K_2^d E_2) &= (r^{-1/3}s^{2/3})^a (r^{1/3}s^{1/3})^b (r^{-1/3}s^{2/3})^{2c} \\
&\quad (r^{1/3}s^{1/3})^{2d} K_1^{2a+c} K_2^{2b+d} E_1^2 E_2 \\
&= r^{(-a+b-2c+2d)/3} s^{(2a+b+4c+2d)/3} \\
&\quad K_1^{2a+c} K_2^{2b+d} E_1^2 E_2;
\end{aligned}$$

and

$$\begin{aligned}
(K_1^a K_2^b E_1)(K_1^c K_2^d E_2)(K_1^a K_2^b E_1) &= (r^{-1/3}s^{2/3})^c (r^{1/3}s^{1/3})^d (r^{-1/3}s^{-1/3})^a \\
&\quad (r^{-1/3}s^{2/3})^a (r^{-2/3}s^{1/3})^b (r^{1/3}s^{1/3})^b \\
&\quad K_1^{2a+c} K_2^{2b+d} E_1 E_2 E_1 \\
&= r^{(-2a-b-c+d)/3} s^{(a+2b+2c+d)/3} \\
&\quad K_1^{2a+c} K_2^{2b+d} E_1 E_2 E_1;
\end{aligned}$$

and

$$\begin{aligned}(K_1^cK_2^dE_2)(K_1^aK_2^bE_1)(K_1^aK_2^bE_1) &= (r^{-1/3}s^{-1/3})^a(r^{-2/3}s^{1/3})^b(r^{-2/3}s^{1/3})^a\\ &\quad (r^{-1/3}s^{2/3})^bK_1^{2a+c}K_2^{2b+d}E_2E_1^2\\ &= r^{(-3a-3b)/3}s^{3b/3}K_1^{2a+c}\\ &\quad K_2^{2b+d}E_2E_1^2.\end{aligned}$$

Applying the automorphism θ to the first two-parameter quantum Serre relation

$$E_1^2E_2-(r+s)E_1E_2E_1+rsE_2E_1^2=0,$$

we have the following system of equations

$$\begin{aligned}-a+b-2c+2d &= -2a-b-c+d;\\ -3a-3b &= -2a-b-c+d;\\ 2a+b+4c+2d &= a+2b+2c+d;\\ 3b &= a+2b+2c+d.\end{aligned}$$

It is easy to see that the previous system of equations is reduced to the following system of equations

$$\begin{aligned}a+2b-c+d &= 0;\\ a-b+2c+d &= 0.\end{aligned}$$

Similarly, from the second two-parameter quantum Serre relation

$$E_1E_2^2-(r+s)E_2E_1E_2+rsE_2^2E_1=0,$$

we also have the same system of equations as follows

$$\begin{aligned}a+2b-c+d &= 0;\\ a-b+2c+d &= 0.\end{aligned}$$

Solving the system

$$\begin{aligned}a+2b-c+d &= 0;\\ a-b+2c+d &= 0;\end{aligned}$$

we have that $b=c$ and $a+b+d=0$. Thus we have proved the theorem as desired. □

2.2. **Hopf algebra automorphisms of $\check{U}_{r,s}^{\geq 0}(\mathfrak{s}l_3)$.** In this subsection, we further determine all the Hopf algebra automorphisms of the Hopf algebra $\check{U}_{r,s}^{\geq 0}(\mathfrak{s}l_3)$. Let us denote by $Aut_{Hopf}(\check{U}_{r,s}^{\geq 0}(\mathfrak{s}l_3))$ the group of all Hopf algebra automorphisms of $\check{U}_{r,s}^{\geq 0}(\mathfrak{s}l_3)$.

First of all, we have the following result

Theorem 2.2. *Let $\theta \in Aut_{Hopf}(\check{U}^{\geq 0}_{r,s}(\mathfrak{sl}_3))$. Then for $l = 1, 2$, we have the following*

$$\theta(K_l) = K_l, \quad \theta(E_l) = b_l E_l,$$

where $b_l \in \mathbb{C}^*$. In particular, we have

$$Aut_{Hopf}(\check{U}^{\geq 0}_{r,s}(\mathfrak{sl}_3)) \cong (\mathbb{C}^*)^2.$$

Proof: Let $\theta \in Aut_{Hopf}(\check{U}^{\geq 0}_{r,s}(\mathfrak{sl}_3))$ be a Hopf algebra automorphism of $\check{U}^{\geq 0}_{r,s}(\mathfrak{sl}_3)$, then we have $\theta \in Aut_{\mathbb{C}}(\check{U}^{\geq 0}_{r,s}(\mathfrak{sl}_3))$. Therefore, we have the following

$$\begin{aligned} \theta(K_l) &= a_l K_l; \\ \theta(E_1) &= b_1 K_1^a K_2^b E_1; \\ \theta(E_2) &= b_2 K_1^c K_2^d E_2; \end{aligned}$$

where $a_l, b_l \in \mathbb{C}^*$ for $l = 1, 2$ and $a, b, c, d \in \mathbb{Z}$ such that $b = c$ and $a + b + d = 0$.

First of all, we need to prove that $a_l = 1$ for $l = 1, 2$. Since θ is a Hopf algebra automorphism, we have the following

$$(\theta \otimes \theta)(\Delta(K_l)) = \Delta(\theta(K_l))$$

for $l = 1, 2$, which imply the following

$$a_l^2 = a_l$$

for $l = 1, 2$. Thus we have $a_l = 1$ for $l = 1, 2$ as desired.

Second of all, we need to prove that $a = b = c = d = 0$. Note that we have the following

$$\begin{aligned} \Delta(\theta(E_1)) &= \Delta(b_1 K_1^a K_2^b E_1) \\ &= \Delta(b_1 K_1^a K_2^b)\Delta(E_1) \\ &= b_1(K_1^a K_2^b \otimes K_1^a K_2^b)(E_1 \otimes 1 + K_1^2 K_2^{-1}) \\ &= b_1 K_1^a K_2^b E_1 \otimes K_1^a K_2^b + b_1 K_1^a K_2^b K_1^2 K_2^{-2} \otimes K_1^a K_2^b E_1 \\ &= \theta(E_1) \otimes K_1^a K_2^b + K_1^a K_2^b K_1^2 K_2^{-1} \otimes \theta(E_1) \end{aligned}$$

and

$$\begin{aligned} (\theta \otimes \theta)(\Delta(E_1)) &= (\theta \otimes \theta)(E_1 \otimes 1 + K_1^2 K_2^{-1} \otimes E_1) \\ &= \theta(E_1) \otimes 1 + \theta(K_1^2 K_2^{-1}) \otimes \theta(E_1) \\ &= \theta(E_1) \otimes 1 + K_1^2 K_2^{-1} \otimes \theta(E_1). \end{aligned}$$

Since $\Delta(\theta(E_1)) = (\theta \otimes \theta)\Delta(E_1)$, we have $a = b = 0$. Since $b = c$ and $a + b + d = 0$, we have $a = b = c = d = 0$.

In addition, it is obvious that the algebra automorphism θ defined by $\theta(K_l) = K_l$ and $\theta(E_l) = b_l E_l$ for $l = 1, 2$ is a Hopf algebra automorphism of $\check{U}_{r,s}^{\geq 0}(\mathfrak{s}l_3)$ as well. Therefore, we have proved the theorem. $\square$

3. Derivations and the first Hochschild cohomology group of $U_{r,s}^{+}(\mathfrak{s}l_3)$

In this section, we determine all the derivations of $U_{r,s}^{+}(\mathfrak{s}l_3)$. In particular, we prove each derivation of $U_{r,s}^{+}(\mathfrak{s}l_3)$ can be uniquely written as the sum of an inner derivation and a linear combination to certain specifically defined derivations. As a result, we are able to prove that the first Hochschild cohomology group of $U_{r,s}^{+}(\mathfrak{s}l_3)$ is a two-dimensional vector space over the base field $\mathbb{C}$. All these will be done through an embedding of the algebra $U_{r,s}^{+}(\mathfrak{s}l_3)$ into a quantum torus, whose derivations had been described in [7]. This method has also been successfully used to compute the derivations of the algebra $U_q(\mathfrak{s}l_4^{+})$ in [6].

3.1. **The embedding of $U_{r,s}^{+}(\mathfrak{s}l_3)$ into a quantum torus.** In this subsection, we embed the algebra $U_{r,s}^{+}(\mathfrak{s}l_3)$ into a quantum torus, which enables us to extend the derivations of $U_{r,s}^{+}(\mathfrak{s}l_3)$ to derivations of the quantum torus. Note that the algebra $U_{r,s}^{+}(\mathfrak{s}l_3)$ has a Goldie quotient ring, which we shall denote by $Q(U_{r,s}^{+}(\mathfrak{s}l_3))$. Inside the Goldie quotient ring $Q(U_{r,s}^{+}(\mathfrak{s}l_3))$ of $U_{r,s}^{+}(\mathfrak{s}l_3)$, let us define the following new variables

$$T_1 = E_1, \quad T_2 = E_2 - \frac{1}{r-s} E_3 E_1^{-1}, \quad T_3 = E_3.$$

Concerning the relationships between the variables $T_i, i = 1, 2, 3$, it is easy to see that we have the following proposition

Proposition 3.1. *The following identities hold:*

(1) $T_1 T_2 = s T_2 T_1$;

(2) $T_1 T_3 = r T_3 T_1$;

(3) $T_2 T_3 = r^{-1} T_3 T_2$.

$\square$

Let us denote by A^3 the subalgebra of $Q(U_{r,s}^{+}(\mathfrak{s}l_3))$ generated by $T_1^{\pm 1}, T_2, T_3$, then we have the following

Proposition 3.2. *The subalgebra A^3 is the same as the subalgebra of $Q(U_{r,s}^{+}(\mathfrak{s}l_3))$ generated by $E_1^{\pm 1}, E_2, E_3$. In particular, A^3 is a free module over the subalgebra generated by E_2, E_3.*

□

Furthermore, let us denote by A^2 the subalgebra of $Q(U^+_{r,s}(\mathfrak{s}l_3))$ generated by $T_1^{\pm1}, T_2, T_3^{\pm1}$. Then we have the following proposition

Proposition 3.3. *The subalgebra A^2 is the same as the subalgebra of $Q(U^+_{r,s}(\mathfrak{s}l_3))$ generated by $E_1^{\pm1}, E_2, E_3^{\pm1}$.*

□

Similarly, let us denote by A^1 the subalgebra of $Q(U^+_{r,s}(\mathfrak{s}l_3))$ generated by $T_1^{\pm1}, T_2^{\pm1}, T_3^{\pm1}$. Thanks to **Proposition 3.1**, we know that the indeterminates T_1, T_2, T_3 generate a quantum torus, which we shall denote by $Q_3 = \mathbb{C}_{r,s}[T_1^{\pm1}, T_2^{\pm1}, T_3^{\pm}]$. In particular, we have the following

Proposition 3.4. *The algebra $A^1 = Q_3 = \mathbb{C}_{r,s}[T_1^{\pm1}, T_2^{\pm1}, T_3^{\pm1}]$ is a quantum torus.*

□

Now let us define a linear map

$$\mathcal{I}\colon U^+_{r,s}(\mathfrak{s}l_3) \longrightarrow Q_3$$

from $U^+_{r,s}(\mathfrak{s}l_3)$ into $A^1 = Q_3$ as follows

$$\mathcal{I}(E_1) = T_1, \quad \mathcal{I}(E_2) = T_2 + \frac{1}{r-s}T_3T_1^{-1}, \quad \mathcal{I}(E_3) = T_3.$$

It is easy to see that the linear map $\mathcal{I}$ can be extended to an algebra monomorphism from $U^+_{r,s}(\mathfrak{s}l_3)$ into $A^1 = Q_3$. Furthermore, it is straightforward to prove the following result:

Theorem 3.1. *Let us set $A^4 = U^+_{r,s}(\mathfrak{s}l_3), \Sigma_4 = \{T_1^i \mid i \in \mathbb{Z}_{\geq 0}\}, \Sigma_3 = \{T_3^i \mid i \in \mathbb{Z}_{\geq 0}\}, \Sigma_2 = \{T_2^i \mid i \in \mathbb{Z}_{\geq 0}\}$, then we have the following*

(1) $A^3 = A^4\Sigma_4^{-1}$;

(2) $A^2 = A^3\Sigma_3^{-1}$;

(3) $A^1 = A^2\Sigma_2^{-1}$;

(4) $A^4 \subset A^3 \subset A^2 \subset A^1$;

(5) *The center of A^i is the base field $\mathbb{C}$ for $i = 1, 2, 3, 4$.*

□

From the reference [7], one knows that a derivation D of the quantum torus $A^1 = Q_3$ is of the form $D = ad_t + \delta$ where ad_t is an inner derivation determined by some $t \in A^4$, and δ is a central derivation which acts on the variables $T_i, i = 1, 2, 3$ as follows:

$$\delta(T_i) = \alpha_i T_i$$

for $\alpha_i \in \mathbb{C}$.

Let D be a derivation of $U_{r,s}^+(\mathfrak{s}l_3) = A^4$. According to the previous theorem, one can extend the derivation D to a derivation of A^i for $i = 3, 2, 1$, and thus a derivation of the quantum torus A^1. We still denote the extension by D. Therefore, as a derivation of the quantum torus A^1, the derivation D can be decomposed as follows

$$D = ad_t + \delta$$

where ad_t is an inner derivation determined by some $t \in A^1$, and δ is a central derivation of A^1 which is defined by $\delta(T_i) = \alpha_i T_i$ for $\alpha_i \in \mathbb{C}, i = 1, 2, 3$.

Now we are going to prove that the element t can actually be chosen from the algebra $U_{r,s}^+(\mathfrak{s}l_3)$ and the scalars α_1, α_2 and α_3 are related to each other. In particular, we have the following key lemma.

Lemma 3.1. *The following is true:*

1. *The element t can be chosen from $U_{r,s}^+(\mathfrak{s}l_3)$;*
2. *We have $\alpha_3 = \alpha_1 + \alpha_2$;*
3. *We have $D(E_i) = ad_t(E_i) + \alpha_i E_i$ for $i = 1, 2, 3$.*

Proof: First of all, we show that $t \in A^2$. Suppose that we have $t = \sum_{i,j,k} a_{i,j,k} T_1^i T_3^j T_2^k \in A_1$.

If $k \geq 0$ for all k, then we are done. Otherwise, let us set

$$t_- = \sum_{k<0} a_{i,j,k} T_1^i T_3^j T_2^k$$

and

$$t_+ = \sum_{k\geq 0} a_{i,j,k} T_1^i T_3^j T_2^k.$$

Then we have that $t = t_- + t_+$.

First of all, we have the following

$$\begin{aligned} D(T_1) &= ad_t(T_1) + \delta(T_1) \\ &= (t_- T_1 - T_1 t_-) + (t_+ T_1 - T_1 t_+) + \alpha_1 (T_1) \end{aligned}$$

for some $\alpha_1 \in \mathbb{C}$.

Since D is a derivation of the algebra $U_{r,s}^+(\mathfrak{s}l_3)$ and the variable T_1 is in the algebra $A^4 = U_{r,s}^+(\mathfrak{s}l_3)$, we have that the element $D(T_1)$ is in the algebra A^4, and furthermore in the algebra A^2. Note that the elements of A^2 don't involve negative powers of the variable T_2, thus we have the following

$$t_- T_1 = T_1 t_-.$$

Therefore, we have the following

$$\begin{aligned} T_1(\sum_{k<0} a_{i,j,k} T_1^i T_3^j T_2^k) &= (\sum_{k<0} a_{i,j,k} r^j s^k T_1^i T_3^j T_2^k) T_1 \\ &= (\sum_{k<0} a_{i,j,k} T_1^i T_3^j T_2^k) T_1. \end{aligned}$$

Thus we have $k = 0$. Since we are supposed to have $k < 0$ in the expression of t_-, we have run into a contradiction. Therefore, we have that $t_- = 0$ and $t = t_+ \in A^2$. Similarly, we can also prove that $t \in A^3$.

Since the algebra A^3 is also generated by the elements $E_1^{\pm 1}, E_2, E_3$, we have the following

$$t = \sum_{i,j \geq 0, k \geq 0} a_{i,j,k} E_1^i E_3^j E_2^k.$$

Applying the derivation D to the variable E_3, we can further prove that $i \geq 0$. Therefore, we have proved that $t \in A^4 = U_{r,s}^+(\mathfrak{sl}_3)$ as desired.

Since we have $D = ad_t + \delta$ for some $t \in U_{r,s}^+(\mathfrak{sl}_3)$, we have

$$\begin{aligned} D(E_2) &= ad_t(E_2) + \delta(E_2) \\ &= (tE_2 - E_2 t) + \delta(T_2 + \frac{1}{r-s} T_3 T_1^{-1}) \\ &= (tE_2 - E_2 t) + \alpha_2 (T_2 + \frac{1}{r-s} T_3 T_1^{-1}) \\ &\quad + \frac{1}{r-s}(\alpha_3 - \alpha_1 - \alpha_2) T_3 T_1^{-1} \\ &= (tE_2 - E_2 t) + \alpha_2 E_2 + \frac{1}{r-s}(\alpha_3 - \alpha_1 - \alpha_2) T_3 T_1^{-1}. \end{aligned}$$

Note that $D(E_2) \in U_{r,s}^+(\mathfrak{sl}_3)$, we have $\frac{1}{r-s}(\alpha_3 - \alpha_1 - \alpha_2) T_3 T_1^{-1} \in A^4$. Thus we have the following

$$\alpha_3 = \alpha_1 + \alpha_2$$

and

$$D(E_2)) = ad_t(E_2) + \alpha_2 E_2.$$

So we have the proved the lemma as desired. □

Now let us define two derivations D_1, D_2 of the algebra $U_{r,s}^+(\mathfrak{sl}_3)$ as follows:

$$D_1(E_1) = E_1, \quad D_2(E_2) = 0, \quad D_1(E_3) = E_3;$$
$$D_2(E_1) = 0, \quad D_2(E_2) = E_2, \quad D_2(E_3) = E_3.$$

Based on the previous lemma, we have the following

Theorem 3.2. *Let D be a derivation of $U_{r,s}^+(\mathfrak{sl}_3)$. Then we have*

$$D = ad_t + \mu_1 D_1 + \mu_2 D_2$$

for some $t \in U_{r,s}^+(\mathfrak{sl}_3)$ and $\mu_i \in \mathbb{C}$ for $i = 1, 2$.

□

Recall that the Hochschild cohomology group in degree 1 of $U_{r,s}^+(\mathfrak{sl}_3)$ is denoted by $HH^1(U_{r,s}^+(\mathfrak{sl}_3))$, which is defined as follows

$$HH^1(U_{r,s}^+(\mathfrak{sl}_3)) \colon = Der(U_{r,s}^+(\mathfrak{sl}_3))/InnDer(U_{r,s}^+(\mathfrak{sl}_3)).$$

where $InnDer(U_{r,s}^+(\mathfrak{sl}_3)) \colon = \{ad_t \mid t \in U_{r,s}^+(\mathfrak{sl}_3)\}$ is the Lie algebra of inner derivations of $U_{r,s}^+(\mathfrak{sl}_3)$. It is well known that $HH^1(U_{r,s}^+(\mathfrak{sl}_3))$ is a module over $HH^0(U_{r,s}^+(\mathfrak{sl}_3)) \colon = Z(U_{r,s}^+(\mathfrak{sl}_3)) = \mathbb{C}$.

Now we state the structural result for the first Hochschild cohomology of $U_{r,s}^+(\mathfrak{sl}_3)$.

Theorem 3.3. *The following is true:*

1. *Every derivation D of $U_{r,s}^+(\mathfrak{sl}_3)$ can be uniquely written as follows:*
$$D = ad_t + \mu_1 D_1 + \mu_2 D_2$$
where $ad_t \in InnDer(U_{r,s}^+(\mathfrak{sl}_3))$ and $\mu_1, \mu_2 \in \mathbb{C}$.

2. *The first Hochschild cohomology group $HH^1(U_{r,s}^+(\mathfrak{sl}_3))$ of $U_{r,s}^+(\mathfrak{sl}_3)$ is a two-dimensional vector space spanned by $\overline{D_1}$ and $\overline{D_2}$.*

Proof: Suppose that we have $ad_t + \mu_1 D_1 + \mu_2 D_2 = 0$, then we need to prove that $\mu_1 = \mu_2 = ad_t = 0$. Let us set $\delta = \mu_1 D_1 + \mu_2 D_2$. Then δ is a derivation of the algebra $U_{r,s}^+(\mathfrak{sl}_3)$.

Note that we can extend the derivation δ to a derivation of A^1, and we also have $ad_t + \delta = 0$ as a derivation of A^1. Furthermore, we have the following

$$\delta(T_1) = \mu_1 T_1, \quad \delta(T_2) = \mu_2 T_2, \quad \delta(T_3) = (\mu_1 + \mu_2) T_3.$$

Thus the derivation δ is a central derivation of the quantum torus A^1. According to the result in [7], we have that $ad_t = 0 = \delta$. Thus we have $\mu_1 = \mu_2 = 0$ as desired.

So we have proved the uniqueness for the decomposition of D, which further implies the second statement of the theorem. $\square$

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Department of Mathematics & Computer Science, Fayetteville State University, 1200 Murchison Road, Fayetteville, NC 28301

E-mail address: `xtang@uncfsu.edu`