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Additivity of Lie Centralizers on Triangular Rings

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ABSTRACT. We introduce the definition of Lie centralizers and investigate the additivity of Lie centralizers on triangular rings. Characterizations of centralizers and Lie centralizers on triangular rings and nest algebras are also presented.

1. Introduction and preliminaries

Let $\mathcal{R}$ be a ring. An additive mapping $T : \mathcal{R} \to \mathcal{R}$ is called a *left* (resp. *right*) *centralizer* if $T(ab) = T(a)b$ (resp. $T(ab) = aT(b)$) for any $a, b \in \mathcal{R}$. T is called a *centralizer* if it is both a left and a right centralizer. Centralizers on rings as well as algebras have been extensively investigated by many mathematicians (see [7], [8], [9], [10], [12], [14], and references therein). Motivated by the concept of centralizers on rings, we here introduce the definition of Lie centralizers as follows.

Definition 1.1. Let $\mathcal{R}$ be a ring and $\delta : \mathcal{R} \to \mathcal{R}$ an additive mapping. Then δ is called a *Lie centralizer* of $\mathcal{R}$ if

$$\delta([a, b]) = [\delta(a), b] \quad (\text{or } \delta([a, b]) = [a, \delta(b)])$$

holds true for any $a, b \in \mathcal{R}$, where $[a, b] = ab - ba$ is the usual Lie product of a and b.

Remark 1.2. Observe that if $\delta([a, b]) = [\delta(a), b]$, then we have

$$\delta(ab - ba) = \delta(a)b - b\delta(a).$$

Interchanging a and b in the above identity, we have

$$\delta(ba - ab) = \delta(b)a - a\delta(b).$$

Replacing a with $-a$ in the above relation, we arrive at $\delta(ab - ba) = a\delta(b) - \delta(b)a$, which can be written as $\delta([a, b]) = [a, \delta(b)]$. Thus conditions $\delta([a, b]) = [\delta(a), b]$ and $\delta([a, b]) = [a, \delta(b)]$ are equivalent regardless of the additivity of δ.

Recall that an additive map of ring $\mathcal{R}$ into itself is called a *commuting map* if $[T(a), a] = 0$ for arbitrary $a \in \mathcal{R}$.

One can easily check that each centralizer is a Lie centralizer and every Lie centralizer is a commuting map.

Over the last decades a lot of work has been done on the additivity of mappings on rings and operator algebras. We refer the readers to some recent papers [1, 5, 6, 11, 15] where further references can be found. However, to author's knowledge,

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there has no result on additivity of commuting maps on rings. It is the aim of this paper to initiate the study of additivity of commuting maps by investigating the additivity of Lie centralizers on triangular rings. We will show that if δ is a map of a triangular ring $\mathfrak{U} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ to itself satisfying

$$\delta([A, B]) = [\delta(A), B]$$

for any $A, B \in \mathfrak{U}$, then $\delta = \sigma + \tau$, where $\sigma : \mathfrak{U} \to \mathfrak{U}$ is a centralizer and $\tau : \mathfrak{U} \to \mathcal{Z}(\mathfrak{U})$ is a mapping such that $\tau(A+B) = \tau(A)+\tau(B)+Z_{A,B}$ for some $Z_{A,B} \in \mathcal{Z}(\mathfrak{U})$ (depending on A and B) and $\tau([A, B]) = 0$ for any $A, B \in \mathfrak{U}$. Characterizations of centralizers and Lie centralizers on triangular rings and nest algebras are also presented.

Recall that a *triangular ring* $\mathfrak{U} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ (see [2] and [13]) is a ring of the form

$$\mathfrak{U} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} : a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B} \right\}$$

under the usual matrix operations, where $\mathcal{A}$ and $\mathcal{B}$ are two rings over a commutative ring $\mathcal{R}$, and $\mathcal{M}$ is an $(\mathcal{A}, \mathcal{B})$-bimodule which is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module (see [13]). The center of $\mathfrak{U}$ is

$$\mathcal{Z}(\mathfrak{U}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : am = mb \text{ for all } m \in \mathcal{M} \right\}.$$

Let $\pi_{\mathcal{A}} : \mathfrak{U} \to \mathcal{A}$ and $\pi_{\mathcal{B}} : \mathfrak{U} \to \mathcal{B}$ be the natural projections defined by

$$\begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \mapsto a \quad \text{and} \quad \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \mapsto b$$

respectively. Then $\pi_{\mathcal{A}}(\mathcal{Z}(\mathfrak{U})) \subseteq \mathcal{Z}(\mathcal{A})$ and $\pi_{\mathcal{B}}(\mathcal{Z}(\mathfrak{U})) \subseteq \mathcal{Z}(\mathcal{B})$, and there exists a unique ring isomorphism $\omega : \pi_{\mathcal{A}}(\mathcal{Z}(\mathfrak{U})) \to \pi_{\mathcal{B}}(\mathfrak{U}))$ such that $am = m\omega(a)$ for all $m \in \mathcal{M}$ (see [11]).

Let

$$\mathcal{Z}(\mathcal{A})_{\mathfrak{U}} = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in \mathcal{Z}(\mathcal{A}) \right\}$$

and

$$\mathcal{Z}(\mathcal{B})_{\mathfrak{U}} = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} : b \in \mathcal{Z}(\mathcal{B}) \right\}.$$

We set

$$\mathfrak{U}_{11} = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in \mathcal{A} \right\},$$

$$\mathfrak{U}_{12} = \left\{ \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} : m \in \mathcal{M} \right\},$$

and

$$\mathfrak{U}_{22} = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} : b \in \mathcal{B} \right\}.$$

Then we may write $\mathfrak{U} = \mathfrak{U}_{11} \oplus \mathfrak{U}_{12} \oplus \mathfrak{U}_{22}$, and every element $A \in \mathfrak{U}$ can be written as $A = A_{11} + A_{12} + A_{22}$. Note that notation A_{ij} will denote an arbitrary element of $\mathfrak{U}_{ij}$.

Let $\mathcal{H}$ be a Hilbert space over field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. The algebra of all bounded linear operators on $\mathcal{H}$ is denoted by $B(\mathcal{H})$. For $x, y \in \mathcal{H}$, we denote the inner product of these vectors by $\langle x, y\rangle$ and the rank one operator $u \mapsto \langle u, y\rangle x$ by $x \otimes y$. Note that every rank one operator on $\mathcal{H}$ can be written in this form.

2. Additivity of Lie centralizers on triangular rings

In this section, we aim to study the additivity of Lie centralizers on triangular rings. Note that, different from [2], [4], and [15], the rings $\mathcal{A}$ and $\mathcal{B}$ in triangular algebra $\mathfrak{V} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ in most part of this section need not be unital. We also want to mention that our approaches are different from those in [2], [4], and [15] which mainly depend on the existence of identity elements in the underlying rings.

In what follows, $\delta : \mathfrak{V} \to \mathfrak{V}$ will be a mapping such that

$$\delta([A, B]) = [\delta(A), B]$$

holds true for all $A, B \in \mathfrak{V}$, where $\mathfrak{V} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is a triangular ring satisfying

(C1) if $\mathcal{A}m = \{0\}$ (resp. $m\mathcal{B} = \{0\}$), then $m = 0$;

(C2) $\pi_{\mathcal{A}}(\mathcal{Z}(\mathfrak{V})) = \mathcal{Z}(\mathcal{A})$ and $\pi_{\mathcal{B}}(\mathcal{Z}(\mathfrak{V})) = \mathcal{Z}(\mathcal{B})$.

Lemma 2.1. *For any $A, B \in \mathfrak{V}$, there exists a $Z_{A,B} \in \mathcal{Z}(\mathfrak{V})$ such that*

$$\delta(A + B) = \delta(A) + \delta(B) + Z_{A,B}.$$

Proof. For any $A, B, X \in \mathfrak{V}$, noticing that $\delta([A, B]) = [\delta(A), B] = [A, \delta(B)]$, we have

$$\begin{aligned} [\delta(A + B), X] &= [A + B, \delta(X)] \\ &= [A, \delta(X)] + [B, \delta(X)] \\ &= [\delta(A), X] + [\delta(B), X] \\ &= [\delta(A) + \delta(B), X], \end{aligned}$$

which implies that $\delta(A + B) - \delta(A) - \delta(B) \in \mathcal{Z}(\mathcal{R})$. Thus, there exists a $Z_{A,B} \in \mathcal{Z}(\mathfrak{V})$ such that

$$\delta(A + B) = \delta(A) + \delta(B) + Z_{A,B}.$$

□

Lemma 2.2. *(1) $\delta(\mathfrak{V}_{11}) \subseteq \mathfrak{V}_{11} \oplus \mathcal{Z}(\mathcal{B})_{\mathfrak{V}}$.*

(2) $\delta(\mathfrak{V}_{22}) \subseteq \mathcal{Z}(\mathcal{A})_{\mathfrak{V}} \oplus \mathfrak{V}_{22}$.

Proof. We only prove (1). Let $A_{11} \in \mathfrak{V}_{11}$ and write $\delta(A_{11}) = U_{11} + U_{12} + U_{22}$. For any $B_{22} \in \mathfrak{V}_{22}$, we consider

$$\begin{aligned} 0 &= \delta([A_{11}, B_{22}]) \\ &= [\delta(A_{11}), B_{22}] \\ &= \delta(A_{11})B_{22} - B_{22}\delta(A_{11}) \\ &= U_{12}B_{22} + U_{22}B_{22} - B_{22}U_{22}, \end{aligned}$$

which yields that $U_{12}B_{22} = 0$. Therefore, $U_{12} = 0$ and $U_{22}B_{22} = B_{22}U_{22}$. Thus we can conclude that $\delta(A_{11}) = U_{11} + U_{22}$ with $\pi_{\mathcal{B}}(U_{22}) \in \mathcal{Z}(\mathcal{B})$. □

By Lemma 2.2, we can define two mappings $\tau_1 : \mathfrak{V}_{11} \to \mathcal{Z}(\mathfrak{V})$ by

$$\tau_1(A) = \omega^{-1}(\pi_{\mathcal{B}}([\delta(A)]_{22})) \oplus \pi_{\mathcal{B}}([\delta(A)]_{22}) \quad \text{for any} \quad A \in \mathfrak{V}_{11}$$

and $\tau_2 : \mathfrak{V}_{22} \to \mathcal{Z}(\mathfrak{V})$ by

$$\tau_2(B) = \pi_{\mathcal{A}}([\delta(B)]_{11}) \oplus \omega(\pi_{\mathcal{A}}([\delta(B)]_{11})) \quad \text{for any} \quad B \in \mathfrak{V}_{22}.$$

It follows from the proof the above lemma that $\delta(A)-\tau_1(A)\in\mathfrak{V}_{11}$ for any $A\in\mathfrak{V}_{11}$ and $\delta(B)-\tau_2(B)\in\mathfrak{V}_{22}$ for any $B\in\mathfrak{V}_{22}$.

Lemma 2.3. $\delta(\mathfrak{V}_{12})\subseteq\mathfrak{V}_{12}\oplus\mathcal{Z}(\mathfrak{V})$.

Proof. Pick $A_{12}\in\mathfrak{V}_{12}$ and write $\delta(A_{12})=U_{11}+U_{12}+U_{22}$. For any $X_{12}\in\mathfrak{V}_{12}$, we have

$$0=\delta([A_{12},X_{12}])=[\delta(A_{12}),X_{12}]=\delta(A_{12})X_{12}-X_{12}\delta(A_{12})=U_{11}X_{12}-X_{12}U_{22}.$$

It follows that $U_{11}+U_{22}\in\mathcal{Z}(\mathfrak{V})$. □

The above lemma enables us to define a mapping $\tau_3:\mathfrak{V}_{12}\to\mathcal{Z}(\mathfrak{V})$ by

$$\tau_3(A)=[\delta(A)]_{11}+[\delta(A)]_{22}\quad\text{for any}\quad A\in\mathfrak{V}_{12}.$$

It turns out, from the proof of Lemma 2.3, that $\delta(A)-\tau_3(A)\in\mathfrak{V}_{12}$ for any $A\in\mathfrak{V}_{12}$. Suppose that $\tau_3(A)=C_1\in\mathcal{Z}(\mathfrak{V})$ and $\tau_3(A)=C_2\in\mathcal{Z}(\mathfrak{V})$ for $A\in\mathfrak{V}_{12}$. Then

$$C_2-C_1=(\delta(A)-C_1)-(\delta(A)-C_2)\in\mathfrak{V}_{12}\cap\mathcal{Z}(\mathfrak{V})=\{0\}.$$

Thus, τ is well defined.

We now continue to define a mapping $\tau:\mathfrak{V}\to\mathcal{Z}(\mathfrak{V})$ by

$$\tau(A)=\tau_1(A_{11})+\tau_2(A_{22})+\tau_3(A_{12})$$

for any $A=A_{11}+A_{12}+A_{22}\in\mathfrak{V}$ and a mapping $\sigma:\mathfrak{V}\to\mathfrak{V}$ is then defined by

$$\sigma(A)=\delta(A)-\tau(A).$$

By the definitions of σ and τ, one can verify that

Lemma 2.4. *(1) $\sigma(\mathfrak{V}_{ij})\subseteq\mathfrak{V}_{ij}$ for $1\le i\le j\le 2$.*
(2) $\tau(A_{11}+A_{12}+A_{22})=\tau(A_{11})+\tau(A_{12})+\tau(A_{22})$.

Lemma 2.5. *For any $A,B\in\mathfrak{V}$, by Lemma 2.1, there exists a $Z_{A,B}\in\mathcal{Z}(\mathfrak{V})$ such that $\delta(A+B)=\delta(A)+\delta(B)+Z_{A,B}\in\mathcal{Z}(\mathfrak{V})$, then*
(1) $\tau(A+B)=\tau(A)+\tau(B)+Z_{A,B}$.
(2) σ is additive.

Proof. (1) In view of Lemma 2.4, we only need to consider the following cases:

Case 1. $\tau(A_{12}+B_{12})=\tau(A_{12})+\tau(B_{12})+Z_{A_{12},B_{12}}$ for some $Z_{A_{12},B_{12}}\in\mathcal{Z}(\mathfrak{V})$.
It follows from Lemma 2.1 that there exists a $Z_{A_{12},B_{12}}\in\mathcal{Z}(\mathfrak{V})$ such that

$$\delta(A_{12}+B_{12})=\delta(A_{12})+\delta(B_{12})+Z_{A_{12},B_{12}}.$$

Therefore,

$$\begin{aligned}&\tau(A_{12}+B_{12})-\tau(A_{12})-\tau(B_{12})-Z_{A_{12},B_{12}}\\=\;&\delta(A_{12}+B_{12})-\sigma(A_{12}+B_{12})-\delta(A_{12})\\&+\sigma(A_{12})-\delta(B_{12})+\sigma(B_{12})-Z_{A_{12},B_{12}}\\=\;&\sigma(A_{12})+\sigma(B_{12})-\sigma(A_{12}+B_{12})\in\mathcal{Z}(\mathfrak{V})\cap\mathfrak{V}_{12}=\{0\}.\end{aligned}$$

Case 2. $\tau(A_{ii}+B_{ii})=\tau(A_{ii})+\tau(B_{ii})+Z_{A_{ii},B_{ii}}$ for some $Z_{A_{ii},B_{ii}}\in\mathcal{Z}(\mathfrak{V})$.
With the same argument as in Case 1, we can get

$$\begin{aligned}&\tau(A_{ii}+B_{ii})-\tau(A_{ii})-\tau(B_{ii})-Z_{A_{ii},B_{ii}}\\=\;&\sigma(A_{ii})+\sigma(B_{ii})-\sigma(A_{ii}+B_{ii})\in\mathcal{Z}(\mathfrak{V})\cap\mathfrak{V}_{ii}=\{0\}.\end{aligned}$$

(2) It is a direct consequence of (1). □

Lemma 2.6. *σ is a centralizer.*

Proof. We divide the proof into three steps.

Step 1. $\sigma(A_{11}B_{12}) = \sigma(A_{11})B_{12}$ and $\sigma(A_{12}B_{22}) = \sigma(A_{12})B_{22}$.
Indeed, by Lemma 2.4,

$$\begin{aligned}\sigma(A_{11}B_{12}) &= \delta(A_{11}B_{12}) = \delta([A_{11}, B_{12}]) \\ &= [\delta(A_{11}), B_{12}] = [\sigma(A_{11}) + \tau(A_{11}), B_{12}] \\ &= [\sigma(A_{11}), B_{12}] = \sigma(A_{11})B_{12} - B_{12}\sigma(A_{11}) \\ &= \sigma(A_{11})B_{12}.\end{aligned}$$

One can check in a straightforward way that $\sigma(A_{12}B_{22}) = \sigma(A_{12})B_{22}$.

Step 2. $\sigma(A_{11}B_{11}) = \sigma(A_{11})B_{11}$ and $\sigma(A_{22}B_{22}) = \sigma(A_{22})B_{22}$.
For arbitrary $X_{12} \in \mathfrak{V}_{12}$, applying Step 1 twice, we have

$$\sigma(A_{11}B_{11})X_{12} = \sigma(A_{11}B_{11}X_{12}) = \sigma(A_{11})B_{11}X_{12},$$

which implies that $[\sigma(A_{11}B_{11}) - \sigma(A_{11})B_{11}]X_{12} = 0$. Since $\mathcal{M}$ is a faithful left $\mathcal{A}$-module, we can conclude that $\sigma(A_{11}B_{11}) = \sigma(A_{11})B_{11}$.

Using the fact that $\mathcal{M}$ is also a faithful right $\mathcal{B}$-module, one can deduce $\sigma(A_{22}B_{22}) = \sigma(A_{22})B_{22}$.

Step 3. $\sigma(AB) = \sigma(A)B$.
We write $A = A_{11} + A_{12} + A_{22}$ and $B = B_{11} + B_{12} + B_{22}$. One one hand, by Lemma 2.5 and Steps 1 and 2 above, we have

$$\begin{aligned}\sigma(AB) &= \sigma[(A_{11} + A_{12} + A_{22})(B_{11} + B_{12} + B_{22})] \\ &= \sigma(A_{11}B_{11} + A_{11}B_{12} + A_{12}B_{22} + A_{22}B_{22}) \\ &= \sigma(A_{11}B_{11}) + \sigma(A_{11}B_{12}) + \sigma(A_{12}B_{22}) + \sigma(A_{22}B_{22}) \\ &= \sigma(A_{11})B_{11} + \sigma(A_{11})B_{12} + \sigma(A_{12})B_{22} + \sigma(A_{22})B_{22}.\end{aligned}$$

On the other hand, applying Lemma 2.4, we get

$$\begin{aligned}\sigma(A)B &= \sigma(A_{11} + A_{12} + A_{22})(B_{11} + B_{12} + B_{22}) \\ &= [\sigma(A_{11}) + \sigma(A_{12}) + \sigma(A_{22})](B_{11} + B_{12} + B_{22}) \\ &= \sigma(A_{11})B_{11} + \sigma(A_{11})B_{12} + \sigma(A_{12})B_{22} + \sigma(A_{22})B_{22}.\end{aligned}$$

Thus, σ is a left centralizer. In a similar manner, one can check that σ is also a right centralizer. □

Lemma 2.7. *For any $A, B \in \mathfrak{V}$, $\tau([A, B]) = 0$.*

Proof. Apply Lemma 2.6, we compute

$$\begin{aligned}\tau([A, B]) &= \delta([A, B]) - \sigma([A, B]) \\ &= [\delta(A), B] - \sigma(AB - BA) \\ &= [\sigma(A) + \tau(A), B] - \sigma(AB) + \sigma(BA) \\ &= [\sigma(A), B] - \sigma(A)B + B\sigma(A) \\ &= 0.\end{aligned}$$

□

Based on the above lemmas, we can conclude that

Theorem 2.8. *Let $\mathfrak{V} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular ring satisfying*
(C1) if $\mathcal{A}m = \{0\}$ (resp. $m\mathcal{B} = \{0\}$), then $m = 0$;
(C2) $\pi_{\mathcal{A}}(\mathcal{Z}(\mathfrak{V})) = \mathcal{Z}(\mathcal{A})$ and $\pi_{\mathcal{B}}(\mathcal{Z}(\mathfrak{V})) = \mathcal{Z}(\mathcal{B})$.
If mapping $\delta : \mathfrak{V} \to \mathfrak{V}$ has the property that

$$\delta([A, B]) = [\delta(A), B]$$

for any $A, B \in \mathfrak{V}$, then $\delta = \sigma + \tau$, where $\sigma : \mathfrak{V} \to \mathfrak{V}$ is a centralizer and $\tau : \mathfrak{V} \to \mathcal{Z}(\mathfrak{V})$ is a mapping such that for any $A, B \in \mathfrak{V}$
(1) $\tau(A + B) = \tau(A) + \tau(B) + Z_{A,B}$ for some $Z_{A,B} \in \mathcal{Z}(\mathfrak{V})$;
(2) $\tau([A, B]) = 0$.

Particularly, we have

Corollary 2.9. *Let $\mathfrak{V} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular ring with the properties:*
(C1) if $\mathcal{A}m = \{0\}$ (resp. $m\mathcal{B} = \{0\}$), then $m = 0$;
(C2) $\pi_{\mathcal{A}}(\mathcal{Z}(\mathfrak{V})) = \mathcal{Z}(\mathcal{A})$ and $\pi_{\mathcal{B}}(\mathcal{Z}(\mathfrak{V})) = \mathcal{Z}(\mathcal{B})$.
Then each Lie centralizer $\delta : \mathfrak{V} \to \mathfrak{V}$ can be expressed as $\delta = \sigma + \tau$, where $\sigma : \mathfrak{V} \to \mathfrak{V}$ is a centralizer and $\tau : \mathfrak{V} \to \mathcal{Z}(\mathfrak{V})$ is an additive mapping that vanishes at commutators.

We now turn our attention to centralizers on triangular rings. Different from the conditions in the above theorems, for a triangular ring $\mathfrak{W} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$, we only assume that both $\mathcal{A}$ and $\mathcal{B}$ are unital. The identity elements of $\mathcal{A}$ and $\mathcal{B}$ are denoted by $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$ respectively. We also use $a \oplus b$ and $\hat{m}$ to denote the elements $\left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & m \\ 0 & 0 \end{smallmatrix}\right]$ respectively. We take the liberty of borrowing Cheung's method ([2]) in the proof of the following theorem.

Theorem 2.10. *Let $\mathfrak{W} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular ring, where both $\mathcal{A}$ and $\mathcal{B}$ are unital. Suppose that mapping $\sigma : \mathfrak{W} \to \mathfrak{W}$ satisfies*

$$\sigma(AB) = \sigma(A)B = A\sigma(B)$$

for all $A, B \in \mathfrak{W}$, then σ is a centralizer and there exists a $T \in \mathcal{Z}(\mathfrak{W})$ with $\pi_{\mathcal{A}}(T) \in \mathcal{Z}(\mathcal{A})$ and $\pi_{\mathcal{B}}(T) \in \mathcal{Z}(\mathcal{B})$ such that for any $A \in \mathfrak{W}$

$$\sigma(A) = TA.$$

Proof. We write

$$\sigma\left(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} f_{11}(a) + g_{11}(m) + h_{11}(b) & f_{12}(a) + g_{12}(m) + h_{12}(b) \\ 0 & f_{22}(a) + g_{22}(m) + h_{22}(b) \end{bmatrix},$$

where $f_{11} : \mathcal{A} \to \mathcal{A}$, $f_{12} : \mathcal{A} \to \mathcal{M}$, $f_{22} : \mathcal{A} \to \mathcal{B}$, $g_{11} : \mathcal{M} \to \mathcal{A}$, $g_{12} : \mathcal{M} \to \mathcal{M}$, $g_{22} : \mathcal{M} \to \mathcal{B}$, $h_{11} : \mathcal{B} \to \mathcal{A}$, $h_{12} : \mathcal{B} \to \mathcal{M}$, and $h_{22} : \mathcal{B} \to \mathcal{B}$ are maps.

Note that $a \oplus 0 = (a \oplus 0)(1_{\mathcal{A}} \oplus 0)$. Then we compute $\sigma(a \oplus 0)$ in two ways:

$$\begin{aligned} \begin{bmatrix} f_{11}(a) & f_{12}(a) \\ 0 & f_{22}(a) \end{bmatrix} &= \sigma(a \oplus 0) = \sigma[(a \oplus 0)(1_{\mathcal{A}} \oplus 0)] = \sigma(a \oplus 0)(1_{\mathcal{A}} \oplus 0) \\ &= \begin{bmatrix} f_{11}(a) & f_{12}(a) \\ 0 & f_{22}(a) \end{bmatrix} \begin{bmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \begin{bmatrix} f_{11}(a) & f_{12}(a) \\ 0 & f_{22}(a) \end{bmatrix} &= \sigma(a \oplus 0) = \sigma[(a \oplus 0)(1_{\mathcal{A}} \oplus 0)] = (a \oplus 0)\sigma(1_{\mathcal{A}} \oplus 0) \\ &= \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f_{11}(1_{\mathcal{A}}) & f_{12}(1_{\mathcal{A}}) \\ 0 & f_{22}(1_{\mathcal{A}}) \end{bmatrix}. \end{aligned}$$

It follows that $f_{12}(a) = f_{22}(a) = 0$ and $f_{11}(a) = af_{11}(1_{\mathcal{A}})$.

In a similar manner by using $a \oplus 0 = (1_{\mathcal{A}} \oplus 0)(a \oplus 0)$, one can easily get $f_{11}(a) = f_{11}(1_{\mathcal{A}})a$. Thus $f_{11}(1_{\mathcal{A}}) \in \mathcal{Z}(\mathcal{A})$.

With the same argument for $0 \oplus b = (0 \oplus b)(0 \oplus 1_{\mathcal{B}}) = (0 \oplus 1_{\mathcal{B}})(0 \oplus b)$ we can deduce that $h_{11}(b) = h_{12}(b) = 0$ and $h_{22}(b) = bh_{22}(1_{\mathcal{B}}) = h_{22}(1_{\mathcal{B}})b$ for any $b \in \mathcal{B}$ and $h_{22}(1_{\mathcal{B}}) \in \mathcal{Z}(\mathcal{B})$.

Using the facts that $\hat{m} = (1_{\mathcal{A}} \oplus 0)\hat{m}$ and $\hat{m} = \hat{m}(0 \oplus 1_{\mathcal{B}})$ we can infer that $g_{11}(m) = g_{12}(m) = 0$ and $g_{12}(m) = f_{11}(1_{\mathcal{A}})m = mh_{22}(1_{\mathcal{B}})$ for any $m \in \mathcal{M}$.

Therefore, σ takes the form

$$\sigma\left(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} f_{11}(1_{\mathcal{A}})a & g_{12}(m) \\ 0 & h_{22}(1_{\mathcal{B}})b \end{bmatrix},$$

with $f_{11}(1_{\mathcal{A}}) \in \mathcal{Z}(\mathcal{A})$, $h_{22}(1_{\mathcal{B}}) \in \mathcal{Z}(\mathcal{B})$, and $g_{12}(m) = f_{11}(1_{\mathcal{A}})m = mh_{22}(1_{\mathcal{B}})$.

Let

$$T = \begin{bmatrix} f_{11}(1_{\mathcal{A}}) & 0 \\ 0 & h_{22}(1_{\mathcal{B}}) \end{bmatrix},$$

then $T \in \mathcal{Z}(\mathfrak{W})$, $\pi_{\mathcal{A}}(T) \in \mathcal{Z}(\mathcal{A})$, $\pi_{\mathcal{B}}(T) \in \mathcal{Z}(\mathcal{B})$, and

$$\sigma\left(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\right) = T\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}.$$

The proof of the theorem is thereby complete. □

Note that if both $\mathcal{A}$ and $\mathcal{B}$ are unital, Condition (1) in Theorem 2.8 is satisfied. Comparing with Theorem 2 in [2], we can have a sufficient condition for each Lie centralizer to be proper as follows.

Corollary 2.11. *Let $\mathfrak{W} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular ring with both $\mathcal{A}$ and $\mathcal{B}$ are unital. If $\pi_{\mathcal{A}}(\mathcal{Z}(\mathfrak{W})) = \mathcal{Z}(\mathcal{A})$ and $\pi_{\mathcal{B}}(\mathcal{Z}(\mathfrak{W})) = \mathcal{Z}(\mathcal{B})$, then each Lie centralizer δ on $\mathfrak{W}$ is proper, that is, there is a $T \in \mathcal{Z}(\mathfrak{W})$ and an additive map $\tau : \mathfrak{W} \to \mathcal{Z}(\mathfrak{W})$ vanishing at each commutator such that*

$$\delta(A) = TA + \tau(A)$$

for all $A \in \mathfrak{W}$.

We complete this paper by considering the nest algebra case. Recall that a nest $\mathcal{N}$ in a Hilbert space $\mathcal{H}$ over the real or complex field $\mathbb{F}$ is a chain of orthogonal projections on $\mathcal{H}$ including 0 and I which is closed in the strong operator topology. The nest algebra associated to $\mathcal{N}$, denoted by $Alg\mathcal{N}$, is the operator algebra consisting of all bounded linear operators that leave $\mathcal{N}$ invariant, i.e.,

$$Alg\mathcal{N} = \{A \in B(\mathcal{H}) : AP = PAP \text{ for all } P \in \mathcal{N}\}.$$

Note that the center of $Alg\mathcal{N}$ is $\mathbb{F}I$, where I is the identity operator on $\mathcal{H}$.

We first look at a very special case.

Lemma 2.12. *Let $\mathcal{H}$ be a Hilbert space $\mathcal{H}$ with $\dim \mathcal{H} > 1$. Suppose that map $\sigma : B(\mathcal{H}) \to B(\mathcal{H})$ satisfies*

$$\sigma(AB) = \sigma(A)B = A\sigma(B)$$

for arbitrary $A, B \in B(\mathcal{H})$. Then σ is automatically linear and there is a $\lambda \in \mathbb{F}$ such that

$$\sigma(A) = \lambda A$$

for all $A \in B(\mathcal{H})$.

Proof. Since $\dim \mathcal{H} > 1$, we may pick vectors $x_0, y_0 \in \mathcal{H}$ with $\langle x_0, y_0 \rangle = 1$ and define a map $T : \mathcal{H} \to \mathcal{H}$ by

$$Tx = \sigma(x \otimes y_0)x_0$$

for arbitrary $x \in \mathcal{H}$.

Claim 1. $\sigma(A) = TA = AT$ for each $A \in B(\mathcal{H})$.

For any $A \in B(\mathcal{H})$, we have

$$\sigma(Ax \otimes y_0) = \sigma(A)x \otimes y_0 = A\sigma(x \otimes y_0).$$

Applying this equation to x_0, we arrive at

$$TAx = \sigma(A)x = ATx$$

for all $x \in \mathcal{H}$. Consequently, $TA = \sigma(A) = AT$.

Claim 2. T is linear.

By the definition of T, for any $x, y \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{F}$, we have

$$\begin{aligned} T(\alpha x + \beta y) &= \sigma((\alpha x + \beta y) \otimes y_0)x_0 \\ &= \sigma((\alpha x + \beta y) \otimes y_0 \cdot x_0 \otimes y_0)x_0 \\ &= (\alpha x + \beta y) \otimes y_0 \sigma(x_0 \otimes y_0)x_0 \\ &= \alpha x \otimes y_0 \sigma(x_0 \otimes y_0)x_0 + \beta y \otimes y_0 \sigma(x_0 \otimes y_0)x_0 \\ &= \alpha\sigma(x \otimes y_0 \cdot x_0 \otimes y_0)x_0 + \beta\sigma(y \otimes y_0 \cdot x_0 \otimes y_0)x_0 \\ &= \alpha\sigma(x \otimes y_0)x_0 + \beta\sigma(y \otimes y_0)x_0 \\ &= \alpha Tx + \beta Ty. \end{aligned}$$

Claim 3. T is bounded.

Let $\{x_n\} \subseteq \mathcal{H}$ be an arbitrary sequence with $x_n \to x$ and $Tx_n \to y$ as $n \to \infty$. Using the fact that $x_n \otimes y_0 \to x \otimes y_0$ as $n \to \infty$, we have

$$\begin{aligned} y &= \lim_{n\to\infty} Tx_n \\ &= \lim_{n\to\infty} \sigma(x_n \otimes y_0)x_0 \\ &= \lim_{n\to\infty} \sigma(x_n \otimes y_0 \cdot x_0 \otimes y_0)x_0 \\ &= \lim_{n\to\infty} x_n \otimes y_0 \sigma(x_0 \otimes y_0)x_0 \\ &= x \otimes y_0 \sigma(x_0 \otimes y_0)x_0 \\ &= \sigma(x \otimes y_0 \cdot x_0 \otimes y_0)x_0 \\ &= \sigma(x \otimes y_0)x_0 \\ &= Tx. \end{aligned}$$

By Closed Graph Theorem, we can infer that T is bounded.

Claim 4. $T = \lambda I$ for some $\lambda \in \mathbb{F}$.

Note that the center of $B(\mathcal{H})$ is $\mathbb{F}I$. Now, it follows from Claims 1-3 that there exists a scalar $\lambda \in \mathbb{F}$ such that $T = \lambda I$.

□

Theorem 2.13. *Let $Alg\mathcal{N}$ be a nest algebra in a Hilbert space $\mathcal{H}$ with $\dim \mathcal{H} > 1$. Suppose that map $\sigma : Alg\mathcal{N} \to Alg\mathcal{N}$ satisfies*

$$\sigma(AB) = \sigma(A)B = A\sigma(B)$$

for any $A, B \in Alg\mathcal{N}$, then σ is of the form $\sigma(A) = \lambda A$ for some $\lambda \in \mathbb{F}$. Moreover, σ is automatically linear.

Proof. Case 1. $\mathcal{N}$ is a trivial nest; that is $\mathcal{N} = \{0, I\}$. Then $Alg\mathcal{N} = B(\mathcal{H})$. It follows directly from Lemma 2.12.

Case 2. $\mathcal{N}$ is nontrivial, i.e., there is $P \in \mathcal{N}\backslash\{0, I\}$. Then by [3], $Alg\mathcal{N}$ can be viewed as a triangular algebra

$$Alg\mathcal{N} = \left[\begin{array}{cc} Alg(P\mathcal{N}P) & P(Alg\mathcal{N})(I-P) \\ 0 & Alg((I-P)\mathcal{N}(I-P)) \end{array} \right].$$

Thus, by Theorem 2.10, there exist a $\lambda \in \mathbb{F}$ such that $\sigma(A) = \lambda A$ for any $A \in Alg\mathcal{N}$ since $\mathcal{Z}(Alg\mathcal{N}) = \mathbb{F}I$. Now, the automatic linearity of σ is obvious. □

In particular, we have

Corollary 2.14. *Let $Alg\mathcal{N}$ be a nest algebra in a Hilbert space $\mathcal{H}$ over $\mathbb{F}$ with $\dim \mathcal{H} > 1$. If $\sigma : Alg\mathcal{N} \to Alg\mathcal{H}$ is a linear centralizer, then there exists a scalar $\lambda \in \mathbb{F}$ such that $\sigma(A) = \lambda A$ for all $A \in Alg\mathcal{N}$.*

As for Lie centralizers on nest algebras, we have the following results.

Corollary 2.15. *Let $Alg\mathcal{N}$ be a nest algebra in a Hilbert space $\mathcal{H}$ with $\dim \mathcal{H} > 1$. Suppose that map $\delta : Alg\mathcal{N} \to Alg\mathcal{N}$ satisfies*

$$\delta([A, B]) = [\delta(A), B]$$

for any $A, B \in Alg\mathcal{N}$, then δ is of the form $\delta(A) = \lambda A + \tau(A)I$, where $\lambda \in \mathbb{F}$ and $\tau : Alg\mathcal{N} \to \mathbb{F}$ is a mapping satisfying $\tau(A + B) = \tau(A) + \tau(B) + \lambda_{A,B}I$ for some $\lambda_{A,B} \in \mathbb{F}$ and $\tau([A, B]) = 0$ for any $A, B \in Alg\mathcal{N}$.

Corollary 2.16. *Let $Alg\mathcal{N}$ be a nest algebra in a Hilbert space $\mathcal{H}$ over $\mathbb{F}$ with $\dim \mathcal{H} > 1$. If $\delta : Alg\mathcal{N} \to Alg\mathcal{N}$ is a linear Lie centralizer, then $\delta(A) = \lambda A + \tau(A)I$, where $\lambda \in \mathbb{F}$ and $\tau : Alg\mathcal{N} \to \mathbb{F}$ is a linear mapping sending commutators to zero.*

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