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# DERIVATIONS OF THE TWO-PARAMETER QUANTIZED ENVELOPING ALGEBRA $U_{r,s}^+(B_2)$

XIN TANG

ABSTRACT. Let  $r, s$  be two parameters chosen from  $\mathbb{C}^*$  such that  $r^m s^n = 1$  implies  $m = n = 0$ . We compute the derivations of the two-parameter quantized enveloping algebra  $U_{r,s}^+(B_2)$  and calculate its first degree Hochschild cohomology group. We further determine the group of algebra automorphisms for the two-parameter Hopf algebra  $\check{U}_{r,s}^{\geq 0}(B_2)$ . As a result, we determine the group of Hopf algebra automorphisms for  $\check{U}_{r,s}^{\geq 0}(B_2)$ .

## INTRODUCTION

Let  $\mathfrak{g}$  be a finite dimensional complex simple Lie algebra. The two-parameter quantized enveloping algebras ( or quantum groups)  $U_{r,s}(\mathfrak{g})$  have been studied by Benkart and Witherspoon in the references [5, 6] for Lie algebras of type  $A$ . The two-parameter quantized enveloping algebras  $U_{r,s}(\mathfrak{g})$  have been further studied for Lie algebras  $\mathfrak{g}$  of type  $B, C, D$  in [1]. Overall, the two-parameter quantized enveloping algebras  $U_{r,s}(\mathfrak{g})$  are close analogues of their one-parameter peers, and share a similar structural and representation theory as the one-parameter quantized enveloping algebras  $U_q(\mathfrak{g})$ . For instance, the positive part of the two-parameter quantized enveloping algebra  $U_{r,s}(\mathfrak{g}^+)$  is also proved to be isomorphic to certain two-parameter Ringel-Hall algebra in [10].

Nonetheless, there are some differences between the one-parameter quantized enveloping algebras and two-parameter quantized enveloping algebras. In particular, the center of two-parameter quantized enveloping algebras  $U_{r,s}(\mathfrak{g})$  already posed a different picture [2]. Besides, it seems that the two-parameter quantized enveloping algebras  $U_{r,s}(\mathfrak{g})$  are more rigid in that they possess less symmetry. On the one hand, these differences make it more interesting to further investigate these

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algebras. On the other hand, these differences also make it plausible to more effectively study the structures of these algebras.

Recently, there have been some interests in the study of the subalgebras of two-parameter quantized enveloping algebras. For example, both the derivations of the subalgebra  $U_{r,s}^+(sl_3)$  and the automorphisms of augmented Hopf algebra  $\check{U}_{r,s}^{\geq 0}(sl_3)$  have been determined in [11]. In this paper, we will study the derivations of the algebra  $U_{r,s}^+(B_2)$  and the automorphisms of the Hopf algebra  $\check{U}_{r,s}^{\geq 0}(B_2)$ . First of all, we will completely determine the derivations and calculate the first degree Hochschild cohomology group for the algebra  $U_{r,s}^+(B_2)$ . Second of all, we will determine both the algebra automorphism group and Hopf algebra automorphism group of the Hopf algebra  $\check{U}_{r,s}^{\geq 0}(B_2)$ . To calculate the derivations, one needs to embed the algebra  $U_{r,s}^+(B_2)$  into a quantum torus, where the derivations are known [9]. Via this embedding, we shall be able to pull the information on derivations back to the algebra  $U_{r,s}^+(B_2)$ . As a matter of fact, applying a result in [9], we will be able to determine all the derivations of the algebra  $U_{r,s}^+(B_2)$  up to its inner derivations. As a result, we show that the first degree Hochschild cohomology group  $HH^1(U_{r,s}^+(B_2))$  of  $U_{r,s}^+(B_2)$  is indeed a 2-dimensional vector space over the base field  $\mathbb{C}$ . In order to determine the automorphisms of  $\check{U}_{r,s}^{\geq 0}(B_2)$ , we will follow the lines in [7]; and we obtain similar results as those in [7, 11].

The paper is organized as follows. In Section 1, we recall some basic definitions and properties on the two-parameter quantized enveloping algebras  $U_{r,s}^+(B_2)$ , and establish some necessary commuting identities; then we determine the derivations and calculate the first degree Hochschild cohomology group. In Section 2, we first determine the algebra automorphism group of  $\check{U}_{r,s}^{\geq 0}(B_2)$ ; and then we determine the Hopf algebra automorphism group of  $\check{U}_{r,s}^{\geq 0}(B_2)$ .

## 1. DERIVATIONS AND THE FIRST DEGREE HOCHSCHILD COHOMOLOGY GROUP OF $U_{r,s}^+(B_2)$

In this section, we compute all the derivations of the two-parameter quantized enveloping algebra  $U_{r,s}^+(B_2)$ . As a matter of fact, we are able to show that every derivation of  $U_{r,s}^+(B_2)$  can be uniquely decomposed as the sum of an inner derivation plus a linear combination of certain specifically defined derivations. As a result, we prove that the first degree Hochschild cohomology group of  $U_{r,s}^+(B_2)$  is a two-dimensional vector space over the center of  $U_{r,s}^+(B_2)$ , which can be proved to be the base field  $\mathbb{C}$ . The computation of derivations will be carried out

via an embedding of the algebra  $U_{r,s}^+(B_2)$  into a quantum torus, whose derivations had been described in [9]; and this embedding allows to pull information on derivations of the quantum torus back to the algebra  $U_{r,s}^+(B_2)$ . We should mention that this method has also been successfully used to compute the derivations of some quantum algebras such as  $U_q(\mathfrak{sl}_4^+)$  in [8] and  $U_{r,s}^+(sl_3)$  in [11]. In addition, the derivations of  $U_q^+(B_2)$  were determined in [3] via a similar approach.

**1.1. Some basic properties of the algebra  $U_{r,s}^+(B_2)$ .** Recall that two-parameter quantum groups  $U_{r,s}(B_n)$  associated to the complex simple Lie algebras of type  $B_n, n \geq 2$  have been studied in [1]. For our convenience, we will recall the construction for one of the subalgebras, i.e., the algebra  $U_{r,s}^+(B_2)$  in the case  $B_2$  here. From now on, we will always assume that the parameters  $r, s$  are chosen from  $\mathbb{C}^*$  such that  $r^m s^n = 1$  implies  $m = n = 0$ .

First of all, we recall the following definition from the reference [1]:

**Definition 1.1** (See [1]). The two-parameter quantized enveloping algebra  $U_{r,s}^+(B_2)$  is defined to be the  $\mathbb{C}$ -algebra generated by the generators  $e_1, e_2$  subject to the following relations:

$$\begin{aligned} e_1^2 e_2 - (r^2 + s^2) e_1 e_2 e_1 + r^2 s^2 e_2 e_1^2 &= 0, \\ e_1 e_2^3 - (r^2 + rs + s^2) e_2 e_1 e_2^2 + rs(r^2 + rs + s^2) e_2^2 e_1 e_2 - r^3 s^3 e_2^3 e_1 &= 0. \end{aligned}$$

In the rest of this subsection, we will establish some basic properties of the algebra  $U_{r,s}^+(B_2)$ . In particular, we will show that the two-parameter quantized enveloping algebra  $U_{r,s}^+(B_2)$  can be presented as an iterated skew polynomial ring. As a result, we will be able construct a PBW-basis for  $U_{r,s}^+(B_2)$ , and prove that the center of  $U_{r,s}^+(B_2)$  is reduced to  $\mathbb{C}$ .

Now we fix some notation by setting the following new variables  $X_1, X_2, X_3$  and  $X_4$ :

$$\begin{aligned} X_1 &= e_1, & X_2 &= e_3 = e_1 e_2 - r^2 e_2 e_1, \\ X_3 &= e_2 e_3 - s^{-2} e_3 e_2, & X_4 &= e_2. \end{aligned}$$

Concerning the relations between these new variables, we shall have the following lemma:

**Lemma 1.1.** *The following identities hold:*

- (1)  $X_1 X_2 = s^2 X_2 X_1$ ;
- (2)  $X_1 X_3 = r^2 s^2 X_3 X_1$ ;

$$(3) \quad X_2X_3 = rsX_3X_2;$$

$$(4) \quad X_1X_4 = r^2X_4X_1 + X_2;$$

$$(5) \quad X_2X_4 = s^2X_4X_2 - s^2X_3;$$

$$(6) \quad X_4X_3 = r^{-1}s^{-1}X_3X_4.$$

**Proof:** These identities can be verified via straightforward computation and we will skip the details here.  $\square$

In addition, let us define some algebra automorphisms  $\tau_2, \tau_3$ , and  $\tau_4$ , and some derivations  $\delta_2, \delta_3$ , and  $\delta_4$  as follows:

$$\begin{aligned} \tau_2(X_1) &= s^{-2}X_1, & \delta_2(X_2) &= 0, \\ \tau_3(X_1) &= r^{-2}s^{-2}X_1, & \tau_3(X_2) &= r^{-1}s^{-1}X_2, \\ & & \delta_3(X_1) &= 0, \delta_3(X_2) = 0, \\ \tau_4(X_1) &= r^{-2}X_1, & \tau_4(X_2) &= S^{-2}X_2, \tau_4(X_3) = r^{-1}s^{-1}X_3, \\ & & \delta_4(X_1) &= -r^{-1}X_2, \delta_4(X_2) = X_3, \delta_4(X_3) = 0. \end{aligned}$$

$\square$

Based on the previous lemma, it is easy to see that we have the following result

**Theorem 1.1.** *The algebra  $U_{r,s}^+(B_2)$  can be presented as an iterated skew polynomial ring. In particular, we have the following result*

$$U_{r,s}^+(B_2) \cong \mathbb{C}[X_1][X_2, \tau_2, \delta_2][X_3, \tau_3, \delta_3][X_4, \tau_4, \delta_4].$$

$\square$

Based on the previous theorem, we have an obvious corollary as follows:

**Corollary 1.1.** *The set*

$$\{X_1^a X_2^b X_3^c X_4^d \mid a, b, c, d \in \mathbb{Z}_{\geq 0}\}$$

*forms a PBW-basis of the algebra  $U_{r,s}^+(B_2)$ . In particular,  $U_{r,s}^+(B_2)$  has a GK-dimension of 4.*

$\square$

Associated to this iterated skew polynomial ring presentation of  $U_{r,s}^+(B_2)$ , one can define a filtration of  $U_{r,s}^+(B_2)$  such that the corresponding graded algebra  $grU_{r,s}^+(B_2)$  is a quantum space generated by

the variables  $\overline{X_1}, \overline{X_2}, \overline{X_3}$  and  $\overline{X_4}$  subject to the following relations:

$$\begin{aligned}\overline{X_1} \overline{X_2} &= s^2 \overline{X_2} \overline{X_1}; \\ \overline{X_1} \overline{X_3} &= r^2 s^2 \overline{X_3} \overline{X_1}; \\ \overline{X_1} \overline{X_4} &= r^2 \overline{X_4} \overline{X_1}; \\ \overline{X_2} \overline{X_3} &= rs \overline{X_3} \overline{X_2}; \\ \overline{X_2} \overline{X_4} &= s^2 \overline{X_4} \overline{X_2}; \\ \overline{X_3} \overline{X_4} &= rs \overline{X_4} \overline{X_3}.\end{aligned}$$

Now we have the following description of the center of the algebra  $U_{r,s}^+(B_2)$ :

**Corollary 1.2.** *The center of  $U_{r,s}^+(B_2)$  is reduced to the base field  $\mathbb{C}$ .*

**Proof:** Let  $u \in U_{r,s}^+(B_2)$  be an element in the center of  $U_{r,s}^+(B_2)$ . Then  $u$  is a linear combination of the monomials  $X_1^a X_2^b X_3^c X_4^d$ . Then  $\overline{u}$  is in the center of  $grU_{r,s}^+(B_2)$ . Let  $\overline{X_1^a X_2^b X_3^c X_4^d}$  be the image of one of these monomials in the graded algebra, then this monomial  $\overline{X_1^a X_2^b X_3^c X_4^d}$  commutes with the generators  $\overline{X_1}, \overline{X_4}$ . Therefore, we have the following

$$\begin{aligned}s^{2b} r^{2c} s^{2c} r^{2d} &= 1; \\ r^{-2a} s^{-2b} (rs)^{-c} &= 1.\end{aligned}$$

Therefore, we have the following

$$\begin{aligned}2b + 2c = 0, \quad 2c + 2d = 0; \\ 2a + c = 0, \quad 2b + c = 0.\end{aligned}$$

Solving this system, we get  $a = b = c = d = 0$ . Therefore, we have  $u \in \mathbb{C}$ .  $\square$

**1.2. The embedding of  $U_{r,s}^+(B_2)$  into a quantum torus.** In this subsection, we construct an embedding of the algebra  $U_{r,s}^+(B_2)$  into a quantum torus. This embedding shall enable us to extend the derivations of  $U_{r,s}^+(B_2)$  to derivations of the quantum torus, and later on pull information backward. The point is that the algebra  $U_{r,s}^+(B_2)$  has a Goldie quotient ring, which we shall denote by  $Q(U_{r,s}^+(B_2))$ . Within the Goldie quotient ring  $Q(U_{r,s}^+(B_2))$  of  $U_{r,s}^+(B_2)$ , let us define the following new variables

$$\begin{aligned}T_1 &= X_1, \quad T_2 = X_2, \\ T_3 &= X_3, \quad T_4 = X_2^{-1} Z' X_1^{-1},\end{aligned}$$

where

$$Z' = (X_1(X_3 + (s^{-2} - r^{-1}s^{-1})X_2X_4) - s^4(X_3 + (s^{-2} - r^{-1}s^{-1})X_2X_4)X_1).$$

Let us set a new variable  $W = X_3 + (s^{-2} - r^{-1}s^{-1})X_2X_4$ , then we have the following lemma:

**Lemma 1.2.** *The following identities hold:*

- (1)  $X_1W = r^2s^2WX_1 + (1 - r^{-1}s)X_2^2$ ;
- (2)  $X_2W = s^2WX_2$ ;
- (3)  $X_3W = WX_3$ ;
- (4)  $X_4W = s^{-2}WX_4$ ;
- (5)  $X_1Z' = r^2s^2Z'X_1$ ;
- (6)  $X_2Z' = Z'X_2$ ;
- (7)  $X_3Z' = r^{-2}s^{-2}Z'X_3$ ;
- (8)  $X_4Z' = r^{-2}s^{-2}Z'X_4$ .

**Proof:** Once again, we can verify these identities by brutal force and will not present the details here.  $\square$

Furthermore, we can easily prove the following proposition, which describes the relations between the variables  $T_1, T_2, T_3$  and  $T_4$ .

**Proposition 1.1.** *The following identities hold:*

- (1)  $T_1T_2 = s^2T_2T_1$ ;
- (2)  $T_1T_3 = r^2s^2T_3T_1$ ;
- (3)  $T_1T_4 = r^2T_4T_2$ ;
- (4)  $T_2T_3 = rsT_3T_2$ ;
- (5)  $T_2T_4 = s^2T_4T_2$ ;
- (6)  $T_3T_4 = rsT_4T_3$ .

$\square$

Now let us denote by  $B^4$  the subalgebra of  $Q(\check{U}_{r,s}^{\geq 0}(B_2))$  generated by  $T_1^{\pm 1}, X_2, X_3, X_4$ , then we have the following

**Proposition 1.2.** *The subalgebra  $B^4$  is the same as the subalgebra of  $Q(\check{U}_{r,s}^{\geq 0}(B_2))$  generated by  $X_1^{\pm 1}, X_2, X_3, X_4$ . In particular,  $B^4$  is a free module over the subalgebra generated by  $X_2, X_3, X_4$ .*

□

Furthermore, let us denote by  $B^3$  the subalgebra of  $Q(\check{U}_{r,s}^{\geq 0}(B_2))$  generated by  $T_1^{\pm 1}, T_2^{\pm 1}, T_3, T_4$ . Then we shall have the following proposition

**Proposition 1.3.** *The subalgebra  $B^3$  is the same as the subalgebra of  $Q(\check{U}_{r,s}^{\geq 0}(B_2))$  generated by  $X_1^{\pm 1}, X_2^{\pm 1}, X_3, X_4$ .*

□

In addition, we will denote by  $B^2$  the subalgebra of  $Q(U_{r,s}^+(B_2))$  generated by  $T_1^{\pm 1}, T_2^{\pm 1}, T_3^{\pm 1}, T_4$ . We denote by  $B^1$  the subalgebra of  $Q(U_{r,s}^+(B_2))$  generated by the variables  $T_1^{\pm 1}, T_2^{\pm 1}, T_3^{\pm 1}, T_4^{\pm 1}$ . It is easy to see that the indeterminates  $T_1, T_2, T_3, T_4$  generate a quantum torus, which we shall denote by

$$Q_4 = \mathbb{C}_{r,s}[T_1^{\pm 1}, T_2^{\pm 1}, T_3^{\pm 1}, T_4^{\pm 1}].$$

In particular, we have the following proposition:

**Proposition 1.4.** *The algebra  $Q_4 = \mathbb{C}_{r,s}[T_1^{\pm 1}, T_2^{\pm 1}, T_3^{\pm 1}, T_4^{\pm 1}]$  is a quantum torus.*

□

Now let us define a linear map

$$\mathcal{I}: U_{r,s}^+(B_2) \longrightarrow Q_4$$

from  $U_{r,s}^+(B_2)$  into  $B^1 = Q_4$ , whose effect on the generators is given as follows

$$\begin{aligned} \mathcal{I}(X_1) &= T_1, & \mathcal{I}(X_2) &= T_2, & \mathcal{I}(X_3) &= T_3, \\ \mathcal{I}(X_4) &= \lambda(T_4 + (s^4 - r^2s^2)T_2^{-1}T_3 + (r^{-1}s - 1)T_2T_1^{-1}) \end{aligned}$$

where  $\lambda = \frac{r}{(r^2 - s^2)(r - s)}$ .

It is easy to see that the linear map  $\mathcal{I}$  can be extended to an algebra monomorphism from  $U_{r,s}^+(B_2)$  into the quantum torus  $B^1 = Q_4$ . Furthermore, it is straightforward to prove the following result:

**Theorem 1.2.** *Let us set  $B^5 = U_{r,s}^+(B_2)$  and the following*

$$\begin{aligned} \Sigma_5 &= \{T_1^i \mid i \in \mathbb{Z}_{\geq 0}\}, & \Sigma_4 &= \{T_2^i \mid i \in \mathbb{Z}_{\geq 0}\}, \\ \Sigma_3 &= \{T_3^i \mid i \in \mathbb{Z}_{\geq 0}\}, & \Sigma_2 &= \{T_4^i \mid i \in \mathbb{Z}_{\geq 0}\}, \end{aligned}$$

then we have the following

$$(1) \quad B^4 = B^5 \Sigma_5^{-1};$$



$$(2) B^3 = B^4 \Sigma_4^{-1};$$

$$(3) B^2 = B^3 \Sigma_3^{-1};$$

$$(4) B^1 = B^2 \Sigma_2^{-1};$$

(5) *The center of  $B^i$  is the base field  $\mathbb{C}$  for  $i = 1, 2, 3, 4, 5$ .*

□

Thanks to the result in [9], one knows that any derivation  $D$  of the quantum torus  $B^1 = Q_4$  is of the form  $D = ad_t + \delta$ , where  $ad_t$  is an inner derivation defined by some element  $t \in B^1$ , and  $\delta$  is a central derivation which acts on the variables  $T_i, i = 1, 2, 3, 4$  as follows:

$$\delta(T_i) = \alpha_i T_i$$

for  $\alpha_i \in \mathbb{C}$ .

Suppose that  $D$  is a derivation of the algebra  $U_{r,s}^+(B_2) = B^5$ . Due to the nature of the algebras  $B^4, B^3, B^2$  and  $B^1$ , we can extend the derivation  $D$  to a derivation of the algebras  $B^4, B^3, B^2$  and  $B^1$  respectively. We will still denote the extended derivations by  $D$ . Since  $B^1 = Q_4$  is a quantum torus and  $D$  is derivation of  $B^1$ , we have the following decomposition

$$D = ad_t + \delta$$

where  $ad_t$  is an inner derivation determined by some element  $t \in B^1$ , and  $\delta$  is a central derivation of  $B^1$ , which is defined by  $\delta(T_i) = \alpha_i T_i$  for  $\alpha_i \in \mathbb{C}, i = 1, 2, 3, 4$ .

We shall prove that the element  $t$  can actually be chosen from the algebra  $U_{r,s}^+(B_2) = B^5$  and the scalars  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$  are somehow related to each other. In particular, we shall prove the following key lemma.

**Lemma 1.3.** *The following statements are true:*

(1) *The element  $t$  can be chosen from  $U_{r,s}^+(B_2)$ ;*

(2) *We have  $\alpha_2 = \alpha_1 + \alpha_4$ ;*

(3) *We have  $\alpha_3 = \alpha_1 + \alpha_4$ ;*

(4) *We have  $D(X_i) = ad_t(X_i) + \alpha_i X_i$  for  $i = 1, 2, 3, 4$ .*

**Proof:** We start the proof by showing that the element  $t$  can actually be chosen from the algebra  $B^2$ . Suppose that we have the following

expression of  $t$  in the algebra  $B^1$ :

$$t = \sum_{i,j,k,l} \lambda_{i,j,k,l} T_1^i T_2^j T_3^k T_4^l.$$

If the index  $l \geq 0$  for all  $l$ , then we have proved  $t \in B^2$ . Otherwise, let us set two elements

$$t_- = \sum_{l < 0} \lambda_{i,j,k,l} T_1^i T_2^j T_3^k T_4^l$$

and

$$t_+ = \sum_{l \geq 0} \lambda_{i,j,k,l} T_1^i T_2^j T_3^k T_4^l.$$

such that we have  $t = t_- + t_+$ .

Since  $D$  is a derivation of the algebra  $B^1$  and  $T_1 \in B^1$ , we can apply the derivation  $D$  to  $T_1$ . And we obtain the following

$$\begin{aligned} D(T_1) &= ad_t(T_1) + \delta(T_1) \\ &= (t_- T_1 - T_1 t_-) + (t_+ T_1 - T_1 t_+) + \alpha_1(T_1) \end{aligned}$$

for some  $\alpha_1 \in \mathbb{C}$ .

Since  $D$  is also regarded as a derivation of the algebra  $U_{r,s}^+(B_2)$  and the variable  $T_1$  is also in the algebra  $B^5 = U_{r,s}^+(B_2)$ , we shall have that the element  $D(T_1)$  is also in the algebra  $B^5$ , and furthermore in the algebra  $B^2$ . Note that all the elements of  $B^2$  don't involve negative powers of the variable  $T_4$ , thus we shall have the following

$$t_- T_1 - T_1 t_- = 0.$$

Therefore, we are supposed to have the following

$$\begin{aligned} T_1 \left( \sum_{l < 0} \lambda_{i,j,k,l} T_1^i T_2^j T_3^k T_4^l \right) &= \left( \sum_{l < 0} \lambda_{i,j,k,l,m,n} r^{2k+2l} s^{2j} T_1^i \right. \\ &\quad \left. T_2^j T_3^k T_4^l \right) T_1 \\ &= \left( \sum_{k < 0} a_{i,j,k,l} T_1^i T_2^j T_3^k T_4^l \right) T_1. \end{aligned}$$

This shows that we have the following equations:

$$\begin{aligned} 2k + 2l &= 0; \\ 2j + 2k &= 0. \end{aligned}$$

In addition, applying  $D$  to  $T_2$ , we can further derive the following equations

$$\begin{aligned} k - 2i &= 0; \\ k + 2l &= 0. \end{aligned}$$

Together, these equations show that we have  $i = j = k = l = 0$ , which is a contradiction. Therefore, we have that  $t^- = 0$ , which implies  $t \in B^2$ . A similar argument can be used to prove that  $t \in B^3$ .

Since the algebra  $B^3$  is also generated by the elements  $T_1^{\pm 1}, T_2^{\pm 1}, X_3, X_4$ , we have the following decomposition of  $t$  in  $B^3$ :

$$t = \sum_{i,j,k \geq 0, l \geq 0} \lambda_{i,j,k,l} T_1^i T_2^j X_3^k X_4^l.$$

Applying the derivation  $D$  to the variable  $T_1 = X_1$ , we can further prove that  $j = l$ , which implies that  $j \geq 0$ . Therefore, we have proved that  $t \in B^4$  as desired. Using a similar argument, we can prove that  $t \in B^5 = U_{r,s}^+(B_2)$  as desired.

Since we have  $D = ad_t + \delta$  for some  $t \in U_{r,s}^+(B_2)$  and some central derivation of  $B^1$ , and  $T_1 = X_1, T_2 = X_2$  and  $T_3 = X_3$ , we have the following

$$D(X_i) = D(T_i) = ad_t T_i + \alpha_i T_i = ad_t X_i + \alpha_i X_i$$

for  $i = 1, 2, 3$ .

In addition, we have the following

$$\begin{aligned} D(X_4) &= ad_t(X_4) + \lambda \delta(T_4 + (s^4 - r^2 s^2) T_2^{-1} T_3 + (r^{-1} s - 1) T_2 T_1^{-1}) \\ &= (tX_4 - X_4 t) + \lambda \alpha_4 T_4 + \lambda (\alpha_3 - \alpha_2) (s^4 - r^2 s^2) T_2^{-1} T_3 \\ &\quad + \lambda (\alpha_2 - \alpha_1) (r^{-1} s - 1) T_2 T_1^{-1} \\ &= (tX_4 - X_4 t) + \alpha_4 X_4 + \lambda (\alpha_3 - \alpha_2 - \alpha_4) (s^4 - r^2 s^2) T_2^{-1} T_3 \\ &\quad + \lambda (\alpha_2 - \alpha_1 - \alpha_4) (r^{-1} s - 1) T_2 T_1^{-1}. \end{aligned}$$

Since the element  $D(X_4)$  is in the algebra  $U_{r,s}^+(B_2)$ , we shall have

$$(\alpha_3 - \alpha_2 - \alpha_4) (s^4 - r^2 s^2) T_2^{-1} T_3 + (\alpha_2 - \alpha_1 - \alpha_4) (r^{-1} s - 1) T_2 T_1^{-1} = 0.$$

Thus we shall have the following  $\alpha_3 = \alpha_1 + 2\alpha_4$  and  $\alpha_2 = \alpha_1 + \alpha_4$ . In particular, we have

$$D(X_4) = ad_t(X_4) + \alpha_4 X_4.$$

So we have proved the lemma as desired.  $\square$

Now let us define two derivations  $D_1, D_2$  of the algebra  $U_{r,s}^+(B_2)$  as follows:

$$\begin{aligned} D_1(X_1) &= X_1, & D_X(X_2) &= X_2, & D_1(X_3) &= X_3, & D_1(X_4) &= 0; \\ D_2(X_1) &= 0, & D_2(X_2) &= X_2, & D_2(X_3) &= X_3, & D_2(X_4) &= X_4. \end{aligned}$$

Thanks to the previous lemma, we can derive the following result

**Theorem 1.3.** *Let  $D$  be a derivation of  $U_{r,s}^+(B_2)$ . Then we have*

$$D = ad_t + \mu_1 D_1 + \mu_2 D_2$$

for some  $t \in U_{r,s}^+(B_2)$  and  $\mu_i \in \mathbb{C}$  for  $i = 1, 2$ .

□

Recall that the Hochschild cohomology group in degree 1 of  $U_{r,s}^+(B_2)$  is denoted by  $HH^1(U_{r,s}^+(B_2))$ , which is defined as follows

$$HH^1(U_{r,s}^+(B_2)) := Der(U_{r,s}^+(B_2)) / InnDer(U_{r,s}^+(B_2)).$$

where  $InnDer(U_{r,s}^+(B_2)) := \{ad_t \mid t \in U_{r,s}^+(B_2)\}$  is the Lie algebra of inner derivations of  $U_{r,s}^+(B_2)$ . And it is well known that  $HH^1(U_{r,s}^+(B_2))$  is a module over  $HH^0(U_{r,s}^+(B_2)) := Z(U_{r,s}^+(B_2)) = \mathbb{C}$ .

We describe the structural of the first degree Hochschild cohomology group of  $U_{r,s}^+(B_2)$  as a vector space over  $\mathbb{C}$ . In particular, we have the following theorem

**Theorem 1.4.** *The following is true:*

- (1) *Every derivation  $D$  of  $U_{r,s}^+(B_2)$  can be uniquely written as follows:*

$$D = ad_t + \mu_1 D_1 + \mu_2 D_2$$

where  $ad_t \in InnDer(U_{r,s}^+(B_2))$  and  $\mu_1, \mu_2 \in \mathbb{C}$ .

- (2) *The first Hochschild cohomology group  $HH^1(U_{r,s}^+(B_2))$  of  $U_{r,s}^+(B_2)$  is a two-dimensional vector space spanned by  $\overline{D_1}$  and  $\overline{D_2}$ .*

**Proof:** Suppose that  $ad_t + \mu_1 D_1 + \mu_2 D_2 = 0$  as a derivation, to finish the proof, we need to show that  $\mu_1 = \mu_2 = ad_t = 0$ . Let us set a derivation  $\delta = \mu_1 D_1 + \mu_2 D_2$ . Then  $\delta$  can be regarded as a derivation of the algebra  $U_{r,s}^+(B_2)$ , which can be further extended to a derivation of  $B^1$ . As a derivation of  $B^1$ , we also have that  $ad_t + \delta = 0$ . Besides, we also have the following

$$\delta(T_1) = \mu_1 T_1, \quad \delta(T_2) = \mu_2 T_2, \quad \delta(T_3) = (\mu_1 + \mu_2) T_3.$$

Thus the derivation  $\delta$  is verified to be a central derivation of the quantum torus  $B^1$ . Therefore, according to the result in [9], we shall have that  $ad_t = 0 = \delta$ . Hence we have  $\mu_1 = \mu_2 = 0$  as desired. This proves the uniqueness of the decomposition of the derivation  $D$ , which further proves the second statement of the theorem. □

## 2. HOPF ALGEBRA AUTOMORPHISMS OF THE HOPF ALGEBRA

$$\check{U}_{r,s}^{\geq 0}(B_2)$$

In this section, we will first determine the algebra automorphism group of the Hopf algebra  $\check{U}_{r,s}^{\geq 0}(B_2)$ . As a result, we are able to determine the Hopf algebra automorphism group of  $\check{U}_{r,s}^{\geq 0}(B_2)$  as well. We will closely follow the approach used in [7]. Note that such an approach has been adopted to investigate the automorphism group of  $\check{U}_{r,s}^{\geq 0}(\mathfrak{sl}_3)$  in [10]. It is no surprise that we derive very similar results to those obtained in [10].

**2.1. The Hopf algebra  $U_{r,s}^{\geq 0}(B_2)$ .** To proceed, we first recall the construction of the Hopf subalgebra of  $U_{r,s}^{\geq 0}(B_2)$  from [1]. Later on, we will define an augmented version of the Hopf algebra  $U_{r,s}^{\geq 0}(B_2)$ .

**Definition 2.1.** The Hopf algebra  $U_{r,s}^{\geq 0}(B_2)$  is defined to be the  $\mathbb{C}$ -algebra generated by the generators  $e_1, e_2$  and  $w_1, w_2$  subject to the following relations:

$$\begin{aligned} w_1 w_1^{-1} &= w_2 w_2^{-1} = 1, & w_1 w_2 &= w_2 w_1; \\ w_1 e_1 &= r^2 s^{-2} e_1 w_1, & w_1 e_2 &= s^2 e_2 w_1; \\ w_2 e_1 &= r^{-2} e_1 w_2, & w_2 e_2 &= r s^{-1} e_2 w_2; \\ e_1^2 e_2 - (r^2 + s^2) e_1 e_2 e_1 + r^2 s^2 e_2 e_1^2 &= 0; \\ e_1 e_2^3 - (r^2 + r s + s^2) e_2 e_1 e_2^2 + r s (r^2 + r s + s^2) e_2^2 e_1 e_2 - r^3 s^3 e_2^3 e_1 &= 0. \end{aligned}$$

It can be easily verified that the following operators define a Hopf algebra structure on  $U_{r,s}^{\geq 0}(B_2)$ .

$$\begin{aligned} \Delta(e_1) &= e_1 \otimes 1 + w_1 \otimes e_1; \\ \Delta(e_2) &= e_2 \otimes 1 + w_2 \otimes e_2; \\ \Delta(w_1) &= w_1 \otimes w_1, & \Delta(w_2) &= w_2 \otimes w_2; \\ S(e_1) &= -w_1 e_1, & S(e_2) &= -w_2 e_2; \\ S(w_1) &= w_1^{-1}, & S(w_2) &= w_2^{-1}; \\ \epsilon(e_1) &= \epsilon(e_2) = 0, & \epsilon(w_1) &= \epsilon(w_2) = 1. \end{aligned}$$

Of course, it is easy to see that we have the following proposition:

**Proposition 2.1.** *The set*

$$\{X_1^a X_2^b X_3^c X_4^d w_1^m w_2^n \mid a, b, c, d \in \mathbb{Z}_{\geq 0}, m, n \in \mathbb{Z}\}$$

*forms a PBW-basis of the algebra  $U_{r,s}^{\geq 0}(B_2)$ .*

□

However, we will not study the algebra  $U_{r,s}^{\geq 0}(B_2)$  in this paper. Instead, in the rest of this paper, we will study the automorphisms of an augmented Hopf algebra  $\check{U}_{r,s}^{\geq 0}(B_2)$ , whose definition will be given in the next subsection.

**2.2. The Augmented Hopf algebra  $\check{U}_{r,s}^{\geq 0}(B_2)$ .** In this subsection, we shall introduce an augmented Hopf algebra  $\check{U}_{r,s}^{\geq 0}(B_2)$ , which contains the algebra  $U_{r,s}^+(B_2)$  as a subalgebra and enlarges the Hopf algebra  $U_{r,s}^{\geq 0}(B_2)$ . To this end, we need to define the following new variables:

$$k_1 = w_1 w_2, \quad k_2 = w_1^{1/2} w_2.$$

It is easy to see that

$$w_1 = k_1^2 k_2^{-2}, \quad w_2 = k_1^{-1} k_2^2.$$

Now we can have the following definition of the algebra  $\check{U}_{r,s}^{\geq 0}(B_2)$ .

**Definition 2.2.** The algebra  $\check{U}_{r,s}^{\geq 0}(\mathfrak{sl}_3)$  is a  $\mathbb{C}$ -algebra generated by  $e_1, e_2, k_1^{\pm 1}$ , and  $k_2^{\pm 1}$  subject to the following relations:

$$\begin{aligned} k_1 k_1^{-1} &= 1 = k_2 k_2^{-1}, & k_1 k_2 &= k_2 k_1; \\ k_1 e_1 &= s^{-2} e_1 k_1, & k_1 e_2 &= r s e_2 k_1; \\ k_2 e_1 &= r^{-1} s^{-1} e_1 k_2, & k_2 e_2 &= r e_2 k_2; \\ e_1^2 e_2 - (r^2 + s^2) e_1 e_2 e_1 + r^2 s^2 e_2 e_1^2 &= 0; \\ e_1 e_2^3 - (r^2 + r s + s^2) e_2 e_1 e_2^2 + r s (r^2 + r s + s^2) e_2^2 e_1 e_2 - r^3 s^3 e_2^3 e_1 &= 0. \end{aligned}$$

□

We now introduce a Hopf algebra structure on  $\check{U}_{r,s}^{\geq 0}(B_2)$  by defining the following operators:

$$\begin{aligned} \Delta(e_1) &= e_1 \otimes 1 + k_1^2 K_2^{-2} \otimes e_1; \\ \Delta(e_2) &= e_2 \otimes 1 + k_1^{-1} K_2^2 \otimes e_2; \\ \Delta(k_1) &= k_1 \otimes k_1, & \Delta(k_2) &= k_2 \otimes k_2; \\ S(e_1) &= -k_1^2 k_2^{-2} e_1, & S(e_2) &= -K_1^{-1} k_2^2 e_2; \\ S(k_1) &= k_1^{-1}, & S(k_2) &= k_2^{-1}; \\ \epsilon(e_1) &= \epsilon(e_2) = 0, & \epsilon(k_1) &= \epsilon(k_2) = 1. \end{aligned}$$

Then it is straightforward to verify the following result:

**Proposition 2.2.** *The algebra  $\check{U}_{r,s}^{\geq 0}(\mathfrak{sl}_3)$  is a Hopf algebra with the co-product, counit and antipode defined as above.*

□

Furthermore, it is easy to see that we have the following result

**Theorem 2.1.** *The Hopf algebra  $\check{U}_{r,s}^{\geq 0}(B_2)$  has a  $\mathbb{C}$ -basis*

$$\{k_1^m k_2^n X_1^a X_2^b X_3^c X_4^d \mid m, n \in \mathbb{Z}, a, b, c, d \in \mathbb{Z}_{\geq 0}\}.$$

□

In particular, one can see that all the invertible elements of  $\check{U}_{r,s}^{\geq 0}(B_2)$  are of the form  $\lambda k_1^m k_2^n$  for some  $\lambda \in \mathbb{C}^*$  and  $m, n \in \mathbb{Z}$ .

**2.3. The algebra automorphism group of  $\check{U}_{r,s}^{\geq 0}(B_2)$ .** Suppose that  $\theta$  denotes an algebra automorphism of the Hopf algebra  $\check{U}_{r,s}^{\geq 0}(B_2)$ . Since  $k_1, k_2$  are invertible elements in the algebra  $\check{U}_{r,s}^{\geq 0}(B_2)$  and  $\theta$  is an algebra automorphism of  $\check{U}_{r,s}^{\geq 0}(B_2)$ , the images  $\theta(k_1), \theta(k_2)$  of  $k_1, k_2$  are invertible elements in  $\check{U}_{r,s}^{\geq 0}(B_2)$ . Recall that the invertible elements of the algebra  $\check{U}_{r,s}^{\geq 0}(B_2)$  are of the form  $\lambda k_1^m k_2^n, \lambda \in \mathbb{C}^*, m, n \in \mathbb{Z}$ . Therefore, we shall have the following expressions of  $\theta(k_1)$  and  $\theta(k_2)$ :

$$\theta(k_1) = \lambda_1 k_1^x k_2^y, \quad \theta(k_2) = \lambda_2 k_1^z k_2^w$$

where  $\lambda_1, \lambda_2 \in \mathbb{C}^*$  and  $x, y, z, w \in \mathbb{Z}$ .

Since  $\theta$  is an algebra automorphism of  $\check{U}_{r,s}^{\geq 0}(B_2)$ , there is an invertible  $2 \times 2$  matrix associated to  $\theta$ . We will denote this matrix by  $M_\theta = (M_{ij})$ . As a matter of fact, we will set the entries  $M_{11} = x, M_{12} = y, M_{21} = z$  and  $M_{22} = w$ . Since  $\theta$  is an algebra automorphism, we know the determinant of  $M_\theta$  is  $\pm 1$ , i.e., we have  $xw - yz = \pm 1$ .

For  $l = 1, 2$ , let us set the following expressions of the images of  $e_1, e_2$  under the automorphism  $\theta$ :

$$\theta(e_l) = \sum_{m_l, n_l, \beta_l^1, \beta_l^2, \beta_l^3, \beta_l^4} \gamma_{m_l, n_l, \beta_l^1, \beta_l^2, \beta_l^3, \beta_l^4} k_1^{m_l} k_2^{n_l} X_1^{\beta_l^1} X_2^{\beta_l^2} X_3^{\beta_l^3} X_4^{\beta_l^4}$$

where  $\gamma_{m_l, n_l, \beta_l^1, \beta_l^2, \beta_l^3, \beta_l^4} \in \mathbb{C}^*$  and  $m_l, n_l \in \mathbb{Z}$  and  $\beta_l^1, \beta_l^2, \beta_l^3, \beta_l^4 \in \mathbb{Z}_{\geq 0}$ .

Then we have the following

**Proposition 2.3.** *Suppose that  $\theta \in \text{Aut}_{\mathbb{C}}(\check{U}_{r,s}^{\geq 0}(B_2))$  is an algebra automorphism of  $\check{U}_{r,s}^{\geq 0}(B_2)$ , then we have  $M_\theta \in GL(2, \mathbb{Z}_{\geq 0})$ .*

**Proof:** Since  $k_1 e_1 = s^{-2} e_1 k_1$  and  $k_2 e_1 = r^{-1} s^{-1} e_1 k_2$  and  $\theta$  is an algebra automorphism of  $\check{U}_{r,s}^{\geq 0}(B_2)$ , we have the following

$$\begin{aligned} \theta(k_1)\theta(e_1) &= s^{-2}\theta(e_1)\theta(k_1); \\ \theta(k_2)\theta(e_1) &= r^{-1}s^{-1}\theta(e_1)\theta(k_2); \end{aligned}$$

which implies that

$$\begin{aligned} & \lambda_1 k_1^x k_2^y \left( \sum_{m_1, n_1, \beta_1^1, \beta_1^2, \beta_1^3, \beta_1^4} \gamma_{m_1, n_1, \beta_1^1, \beta_1^2, \beta_1^3, \beta_1^4} k_1^{m_1} k_2^{n_1} X_1^{\beta_1^1} X_2^{\beta_1^2} X_3^{\beta_1^3} X_4^{\beta_1^4} \right) \\ = & s^{-2} \left( \sum_{m_1, n_1, \beta_1^1, \beta_1^2, \beta_1^3, \beta_1^4} \gamma_{m_1, n_1, \beta_1^1, \beta_1^2, \beta_1^3, \beta_1^4} k_1^{m_1} k_2^{n_1} X_1^{\beta_1^1} X_2^{\beta_1^2} X_3^{\beta_1^3} X_4^{\beta_1^4} \right) \lambda_1 k_1^x k_2^y, \end{aligned}$$

and

$$\begin{aligned} & \lambda_2 k_1^z k_2^w \left( \sum_{m_1, n_1, \beta_1^1, \beta_1^2, \beta_1^3, \beta_1^4} \gamma_{m_1, n_1, \beta_1^1, \beta_1^2, \beta_1^3, \beta_1^4} k_1^{m_1} k_2^{n_1} X_1^{\beta_1^1} X_2^{\beta_1^2} X_3^{\beta_1^3} X_4^{\beta_1^4} \right) \\ = & r^{-1} s^{-1} \left( \sum_{m_1, n_1, \beta_1^1, \beta_1^2, \beta_1^3, \beta_1^4} \gamma_{m_1, n_1, \beta_1^1, \beta_1^2, \beta_1^3, \beta_1^4} k_1^{m_1} k_2^{n_1} X_1^{\beta_1^1} X_2^{\beta_1^2} X_3^{\beta_1^3} X_4^{\beta_1^4} \right) \lambda_2 k_1^z k_2^w. \end{aligned}$$

After the calculations, we shall have the following system of equations:

$$\begin{aligned} (2\beta_1^1 + 2\beta_1^2 + 2\beta_1^3)x + (\beta_1^2 + 2\beta_1^3 + \beta_1^4)y &= 2; \\ (2\beta_1^1 + 2\beta_1^2 + 2\beta_1^3)z + (\beta_1^2 + 2\beta_1^3 + \beta_1^4)w &= 0. \end{aligned}$$

Similarly, we also have the following

$$\begin{aligned} (\beta_2^1 + \beta_2^2 + \beta_2^3)x + (\beta_2^2 + 2\beta_2^3 + \beta_2^4)y &= 0; \\ (\beta_2^1 + \beta_2^2 + \beta_2^3)z + (\beta_2^2 + 2\beta_2^3 + \beta_2^4)w &= 1. \end{aligned}$$

We now define a  $2 \times 2$ -matrix  $B = (b_{ij})$  with the following entries

$$\begin{aligned} b_{11} &= 2\beta_1^1 + 2\beta_1^2 + 2\beta_1^3; \\ b_{21} &= \beta_1^2 + 2\beta_1^3 + \beta_1^4; \\ b_{12} &= \beta_2^1 + \beta_2^2 + \beta_2^3; \\ b_{22} &= \beta_2^2 + 2\beta_2^3 + \beta_2^4. \end{aligned}$$

And we shall have the following system of equations

$$\begin{aligned} b_{11}x + b_{21}y &= 2; \\ b_{12}x + b_{22}y &= 0; \\ b_{11}z + b_{21}w &= 0; \\ b_{12}z + b_{22}w &= 1. \end{aligned}$$

This system of equations implies that we have the following

$$M_\theta B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

which shows that we have

$$M_\theta^{-1} = \begin{pmatrix} b_{11}/2 & b_{12} \\ b_{21}/2 & b_{22} \end{pmatrix}$$



Obviously, we have that  $M_\theta^{-1} = M_{\theta^{-1}}$ , where the matrix  $M_{\theta^{-1}}$  is the corresponding matrix associated to the algebra automorphism  $\theta^{-1}$ . Since the entries  $b_{11}, b_{12}, b_{21}, b_{22}$  are all nonnegative integers, we can conclude that  $M_{\theta^{-1}} \in GL(2, \mathbb{Z}_{\geq 0})$ . Applying the similar arguments to the algebra automorphism  $\theta^{-1}$ , we shall be able to prove that  $M_\theta \in GL(2, \mathbb{Z}_{\geq 0})$  as desired.  $\square$

For the reader's convenience, we recall an important lemma (**Lemma 2.2** from [7]), which characterizes the matrix  $M_\theta$ :

**Lemma 2.1.** *If  $M$  is a matrix in  $GL(n, \mathbb{Z}_{\geq 0})$  such that its inverse matrix  $M^{-1}$  is also in  $GL(n, \mathbb{Z}_{> 0})$ , then we have  $M = (\delta_{i\sigma(j)})_{i,j}$ , where  $\sigma$  is an element of the symmetric group  $\mathbb{S}_n$ .*

$\square$

As a result of **Proposition 2.3** and **Lemma 2.1**, we immediately have the following result which describes the images of  $k_1, k_2$  under an automorphism  $\theta$  of  $\check{U}_{r,s}^{\geq 0}(B_2)$ :

**Corollary 2.1.** *Let  $\theta \in \text{Aut}_{\mathbb{C}}(\check{U}_{r,s}^{\geq 0}(B_2))$  be an algebra automorphism of  $\check{U}_{r,s}^{\geq 0}(B_2)$ . Then for  $l = 1, 2$ , we have*

$$\theta(k_l) = \lambda_l k_{\sigma(l)}$$

where  $\sigma \in \mathbb{S}_2$  and  $\lambda_l \in \mathbb{C}^*$ .

$\square$

Furthermore, we can prove the following result:

**Proposition 2.4.** *Let  $\theta \in \text{Aut}_{\mathbb{C}}(\check{U}_{r,s}^{\geq 0}(B_2))$  be an algebra automorphism of  $\check{U}_{r,s}^{\geq 0}(B_2)$ . Then for  $l = 1, 2$ , we have*

$$\theta(e_l) = \gamma_l k_1^{m_l} k_2^{n_l} e_{\sigma(l)}$$

where  $\gamma_l \in \mathbb{C}^*$  and  $m_l, n_l \in \mathbb{Z}$ .

**Proof:** Let  $\theta \in \text{Aut}_{\mathbb{C}}(\check{U}_{r,s}^{\geq 0}(B_2))$  be an algebra automorphism of  $\check{U}_{r,s}^{\geq 0}(B_2)$ . We will need to consider two cases.

**Case 1:** Suppose that  $\theta(k_1) = \lambda_1 k_1$  and  $\theta(k_2) = \lambda_2 k_2$ , then it suffices to show that we have

$$\theta(e_1) = \gamma_1 k_1^{m_1} k_2^{n_1} e_1, \quad \theta(e_2) = \gamma_2 k_1^{m_2} k_2^{n_2} e_2$$

for some  $\gamma_1, \gamma_2 \in \mathbb{C}^*$  and  $m_1, m_2, n_1, n_2 \in \mathbb{Z}$ .

Note that we have the following relations between  $e_1, e_2$  and  $k_1, k_2$ :

$$k_1 e_1 = s^{-2} e_1 k_1, \quad k_2 e_1 = r^{-1} s^{-1} e_1 k_2.$$

Via applying  $\theta$  to these identities, we shall have the following

$$\begin{aligned}\theta(k_1)\theta(e_1) &= s^{-2}\theta(e_1)\theta(k_1); \\ \theta(K_2)\theta(e_1) &= r^{-1}s^{-1}\theta(e_2)\theta(k_1).\end{aligned}$$

Therefore, we shall the following

$$\begin{aligned}& \lambda_1 k_1 \left( \sum_{m_1, n_1, \beta_1^1, \beta_1^2, \beta_1^3, \beta_1^4} \gamma_{m_1, n_1, \beta_1^1, \beta_1^2, \beta_1^3, \beta_1^4} k_1^{m_1} k_2^{n_1} X_1^{\beta_1^1} X_2^{\beta_1^2} X_3^{\beta_1^3} X_4^{\beta_1^4} \right) \\ &= \lambda_1 s^{-2} \left( \sum_{m_1, n_1, \beta_1^1, \beta_1^2, \beta_1^3} \gamma_{m_1, n_1, \beta_1^1, \beta_1^2, \beta_1^3} K_1^{m_1} k_2^{n_1} X_1^{\beta_1^1} X_2^{\beta_1^2} X_3^{\beta_1^3} X_4^{\beta_1^4} \right) k_1.\end{aligned}$$

In addition, we also have the following

$$\begin{aligned}& \lambda_2 k_2 \left( \sum_{m_1, n_1, \beta_1^1, \beta_1^2, \beta_1^3} \gamma_{m_1, n_1, \beta_1^1, \beta_1^2, \beta_1^3} k_1^{m_1} k_2^{n_1} X_1^{\beta_1^1} X_2^{\beta_1^2} X_3^{\beta_1^3} X_4^{\beta_1^4} \right) \\ &= \lambda_2 r^{-1} s^{-1} \left( \sum_{m_1, n_1, \beta_1^1, \beta_1^2, \beta_1^3} \gamma_{m_1, n_1, \beta_1^1, \beta_1^2, \beta_1^3} K_2^{m_1} K_1^{n_1} X_1^{\beta_1^1} X_2^{\beta_1^2} X_3^{\beta_1^3} X_4^{\beta_1^4} \right) k_2.\end{aligned}$$

Thus we shall have the following

$$\begin{aligned}s^{-(\beta_1^1 + \beta_1^2 + \beta_1^3)} (rs)^{(\beta_1^2 + 2\beta_1^3 + \beta_1^4)} &= s^{-2}, \\ (rs)^{-(\beta_1^1 + \beta_1^2 + \beta_1^3)} r^{(\beta_1^2 + 2\beta_1^3 + \beta_1^4)} &= r^{-1} s^{-1}.\end{aligned}$$

Moreover, the above identities imply the following

$$\begin{aligned}2\beta_1^1 + 2\beta_1^2 + 2\beta_1^3 &= 2; \\ \beta_1^2 + 2\beta_1^3 + \beta_1^4 &= 0; \\ \beta_1^1 + \beta_1^2 + \beta_1^3 &= 1; \\ \beta_1^2 + 2\beta_1^3 + \beta_1^4 &= 0.\end{aligned}$$

Note that all  $\beta_i^j, i, j = 1, 2, 3, 4$  are non-negative integers, thus we shall have that

$$\beta_1^1 = 1, \quad \beta_1^2 = \beta_1^3 = \beta_1^4 = 0.$$

Similarly, we can also verify the following

$$\beta_2^1 = \beta_2^2 = \beta_2^3 = 0, \quad \beta_2^4 = 1.$$

Therefore, we have proved the result for **Case 1**.

**Case 2:** Suppose that  $\theta(k_1) = \lambda_1 k_2$  and  $\theta(k_2) = \lambda_2 k_1$ , we have to prove that  $\theta(e_1) = \gamma_1 k_1^{m_1} k_2^{n_1} e_2$  and  $\theta(e_2) = \gamma_2 k_1^{m_2} k_2^{n_2} e_1$ . We will not repeat the details here because the proof goes the same way as in **Case 1**.  $\square$

Now we are going to verify that, in a sense, the generators  $e_1, e_2$  can not be exchanged by any algebra automorphism  $\theta$  of  $\check{U}_{r,s}^{\geq 0}(B_2)$ . Indeed, we have the following result

**Corollary 2.2.** *Let  $\theta \in \text{Aut}_{\mathbb{C}}(\check{U}_{r,s}^{\geq 0}(B_2))$  be an algebra automorphism of  $\check{U}_{r,s}^{\geq 0}(B_2)$ . Then for  $l = 1, 2$ , we have the following*

$$\theta(k_l) = \lambda_l k_l, \quad \theta(e_l) = \gamma_l k_1^{m_l} k_2^{n_l} e_l$$

where  $\lambda_l, \gamma_l \in \mathbb{C}^*$  and  $m_l, n_l \in \mathbb{Z}$ .

**Proof:** Suppose that  $\theta(k_1) = \lambda_1 k_2$  and  $\theta(e_2) = \gamma_1 k_1^{m_1} k_2^{n_1} e_2$ . Since we have  $\theta(k_1)\theta(e_1) = s^{-2}\theta(e_1)\theta(k_1)$ , we have the following

$$\lambda_1 k_2 \gamma_1 k_1^{m_1} k_2^{m_2} e_2 = s^{-2} \gamma_1 k_1^{m_1} k_2^{m_2} e_2 \lambda_1 k_2.$$

Note that  $k_2 e_2 = r e_2 k_2$ , then we got a contradiction. Therefore, we have proved the statement as desired.  $\square$

The following main theorem describes the algebra automorphism group of the algebra  $\check{U}_{r,s}^{\geq 0}(B_2)$ :

**Theorem 2.2.** *Let  $\theta \in \text{Aut}_{\mathbb{C}}(\check{U}_{r,s}^{\geq 0}(B_2))$  be an algebra automorphism of the algebra  $\check{U}_{r,s}^{\geq 0}(B_2)$ . Then for  $l = 1, 2$ , we have the following*

$$\theta(k_l) = \lambda_l k_l, \quad \theta(e_1) = \gamma_1 k_1^a k_2^b e_1, \quad \theta(e_2) = \gamma_2 k_1^c k_2^d e_2$$

where  $\lambda_l, \gamma_l \in \mathbb{C}^*$  and  $a, b, c, d \in \mathbb{Z}$  such that  $b = 2c, a + 2c + d = 0$ .

**Proof:** Let  $\theta$  be an algebra automorphism of  $\check{U}_{r,s}^{\geq 0}(B_2)$  and suppose that

$$\theta(e_1) = \gamma_1 k_1^a k_2^b e_1, \quad \theta(e_2) = \gamma_2 k_1^c k_2^d e_2.$$

Note that we have the following

$$\begin{aligned} (k_1^a k_2^b e_1)(k_1^a k_2^b e_1)(k_1^c k_2^d e_2) &= (s^2)^a (rs)^b (s^2)^{2c} (rs)^{2d} \\ &\quad k_1^{2a+c} k_2^{2b+d} e_1^2 e_2 \\ &= r^{(b+2d)} s^{(2a+b+4c+2d)} k_1^{2a+c} k_2^{2b+d} e_1^2 e_2; \end{aligned}$$

and

$$\begin{aligned} (k_1^a k_2^b e_1)(k_1^c k_2^d e_2)(k_1^a k_2^b e_1) &= (s^2)^c (rs)^d ((rs)^{-1})^a (s^2)^a (r^{-1})^b (rs)^b \\ &\quad k_1^{2a+c} k_2^{2b+d} e_1 e_2 e_1 \\ &= r^{(d-a)} s^{(a+b+2c+d)} k_1^{2a+c} k_2^{2b+d} e_1 e_2 e_1; \end{aligned}$$

and

$$\begin{aligned} (k_1^c k_2^d e_2)(k_1^a k_2^b e_1)(k_1^a k_2^b e_1) &= (r^{-1} s^{-1})^a (r^{-1})^b (s^2)^a (r^{-1} s^{-1})^a (rs)^b \\ &\quad k_1^{2a+c} k_2^{2b+d} e_2 e_1^2 \\ &= r^{(-2a-b)} s^b k_1^{2a+c} k_2^{2b+d} e_2 e_1^2. \end{aligned}$$

Via applying the automorphism  $\theta$  to the first two-parameter quantum Serre relation

$$e_1^2 e_2 - (r^2 + rs + s^2) e_1 e_2 e_1 + (rs)^2 e_2 e_1^2 = 0$$

we shall have the following system of equations

$$\begin{aligned} b + 2d &= -a + d; \\ -2a - b &= -a + d; \\ 2a + b + 4c + 2d &= a + b + 2c + d; \\ b &= a + b + 2c + d. \end{aligned}$$

It is easy to see that the previous system of equations is reduced to the following system of equations

$$\begin{aligned} a + b + d &= 0; \\ a + 2c + d &= 0. \end{aligned}$$

In addition, direct calculations yield the following

$$\begin{aligned} (k_1^c k_2^d e_2)(k_1^c k_2^d e_2)(k_1^c k_2^d e_2)(k_1^a k_b e_1) &= r^{(-3a-3b-3c-3d)} s^{(-3a-3c)} \\ & k_1^{3c+a} k_2^{3d+b} e_2^3 e_1; \end{aligned}$$

and

$$\begin{aligned} (k_1^c k_2^d e_e)(k_1^c k_2^d e_2)(k_1^a k_2^b e_1)(k_1^c k_2^2 e_2) &= r^{(-2a-2b-3c-3d)} s^{(-2a-c+d)} \\ & k_1^{3c+a} k_2^{3d+b} e_2^2 e_1 e_2; \end{aligned}$$

and

$$\begin{aligned} (k_1^c k_2^d e_2)(k_1^a k_2^b e_1)(k_1^c k_2^d e_2)(k_1^c k_2^d e_2) &= r^{(-a-b-3c-d)} s^{-a+c+2d} \\ & k_1^{3c+a} k_2^{3d+b} e_2 e_1 e_2^2; \end{aligned}$$

and

$$\begin{aligned} (k_1^a k_b e_1)(k_1^c k_2^d e_2)(k_1^c k_2^d e_2)(k_1^c k_2^d e_2) &= r^{(-3c)} s^{(3c+3d)} \\ & k_1^{3c+a} k_2^{3d+b} e_1 e_2^3. \end{aligned}$$

Via applying the automorphism  $\theta$  to the second two-parameter quantum Serre relation

$$\begin{aligned} e_2^3 e_1 - (r^{-2} + r^{-1} s^{-1} + s^{-2}) e_2^2 e_1 e_2 + r^{-1} s^{-1} (r^{-2} + r^{-1} s^{-1} \\ + s^{-2}) e_2 e_1 e_2^2 - (rs)^{-3} e_1 e_2^3 = 0 \end{aligned}$$

we shall have the following system of equations

$$\begin{aligned}
-3a - 3b - 3c - 3d &= -2a - 2b - 3c - 3d; \\
-a - b - 3c - d &= -2a - 2b - 3c - 3d; \\
-3c &= -2a - 2b - 3c - 3d; \\
-3a - 3c &= -2a - c + d; \\
-a + c + 2d &= -2a - c + d; \\
3c + 3d &= -2a - c + d.
\end{aligned}$$

Therefore, we also have the same system of equations as follows

$$\begin{aligned}
a + b + d &= 0; \\
a + 2c + d &= 0.
\end{aligned}$$

Solving the system

$$\begin{aligned}
a + b + d &= 0; \\
a + 2c + d &= 0;
\end{aligned}$$

we have that  $b = 2c$  and  $a + 2c + d = 0$ . Thus we have proved the theorem as desired.  $\square$

**2.4. Hopf algebra automorphisms of  $\check{U}_{r,s}^{\geq 0}(B_2)$ .** In this subsection, we further determine all the Hopf algebra automorphisms of the Hopf algebra  $\check{U}_{r,s}^{\geq 0}(B_2)$ . Let us denote by  $Aut_{Hopf}(\check{U}_{r,s}^{\geq 0}(B_2))$  the group of all Hopf algebra automorphisms of  $\check{U}_{r,s}^{\geq 0}(B_2)$ .

First of all, we have the following result

**Theorem 2.3.** *Let  $\theta \in Aut_{Hopf}(\check{U}_{r,s}^{\geq 0}(B_2))$ . Then for  $l = 1, 2$ , we have the following*

$$\theta(k_l) = k_l, \quad \theta(e_l) = \gamma_l e_l,$$

for some  $\gamma_l \in \mathbb{C}^*$ . In particular, we have

$$Aut_{Hopf}(\check{U}_{r,s}^{\geq 0}(B_2)) \cong (\mathbb{C}^*)^2.$$

**Proof:** First of all, let  $\theta \in Aut_{Hopf}(\check{U}_{r,s}^{\geq 0}(B_2))$  denote a Hopf algebra automorphism of  $\check{U}_{r,s}^{\geq 0}(B_2)$ , then we have  $\theta \in Aut_{\mathbb{C}}(\check{U}_{r,s}^{\geq 0}(B_2))$ . Therefore, we shall have the following

$$\begin{aligned}
\theta(k_l) &= \lambda_l k_l; \\
\theta(e_1) &= \gamma_1 k_1^a k_2^b e_1; \\
\theta(e_2) &= \gamma_2 k_1^c k_2^d e_2;
\end{aligned}$$

for some  $\lambda_l, \gamma_l \in \mathbb{C}^*$  for  $l = 1, 2$ , and  $a, b, c, d \in \mathbb{Z}$  such that  $b = 2c, a + 2c + d = 0$ .

We want to prove that  $\lambda_l = 1$  for  $l = 1, 2$ . Since  $\theta$  is a Hopf algebra automorphism, we shall have the following

$$(\theta \otimes \theta)(\Delta(k_l)) = \Delta(\theta(k_l))$$

for  $l = 1, 2$ , which imply the following

$$\lambda_l^2 = \lambda_l$$

for  $l = 1, 2$ . Therefore, we have  $\lambda_l = 1$  for  $l = 1, 2$ .

Now we need to prove that  $a = b = c = d = 0$ . First of all, note that we have the following

$$\begin{aligned} \Delta(\theta(e_1)) &= \Delta(\gamma_1 k_1^a k_2^b e_1) \\ &= \Delta(\gamma_1 k_1^a k_2^b) \Delta(e_1) \\ &= \gamma_1 (k_1^a k_2^b \otimes k_1^a k_2^b) (e_1 \otimes 1 + k_1^2 k_2^{-2} \otimes e_1) \\ &= \gamma_1 k_1^a k_2^b e_1 \otimes k_1^a k_2^b + \gamma_1 k_1^a k_2^b k_1^2 k_2^{-2} \otimes k_1^a k_2^b e_1 \\ &= \theta(e_1) \otimes k_1^a k_2^b + k_1^a k_2^b k_1^2 k_2^{-2} \otimes \theta(e_1). \end{aligned}$$

Second of all, note that we also have the following

$$\begin{aligned} (\theta \otimes \theta)(\Delta(e_1)) &= (\theta \otimes \theta)(e_1 \otimes 1 + k_1^2 k_2^{-2} \otimes e_1) \\ &= \theta(e_1) \otimes 1 + \theta(k_1^2 k_2^{-2}) \otimes \theta(e_1) \\ &= \theta(e_1) \otimes 1 + k_1^2 k_2^{-2} \otimes \theta(e_1). \end{aligned}$$

Since  $\Delta(\theta(e_1)) = (\theta \otimes \theta)\Delta(e_1)$ , we have  $a = b = 0$ . Since  $b = 2c$  and  $a + 2c + d = 0$ , we have  $a = b = c = d = 0$ .

In addition, it is obvious that the algebra automorphism  $\theta$  defined by  $\theta(k_l) = k_l$  and  $\theta(e_l) = \gamma_l e_l$  for  $l = 1, 2$  is a Hopf algebra automorphism of  $\tilde{U}_{r,s}^{\geq 0}(B_2)$ . Thus, we have proved the theorem.  $\square$

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## REFERENCES

- [1] Bergeron, N., Gao, Y. and Hu, N., Drinfeld doubles and Lusztig's symmetries of two-parameter quantum groups, *J. Algebra* **301** (2006), no. 1, 378–405.
- [2] Benkart, G., Kang, S.J and Lee K.H., On the center of two-parameter quantum groups, *Roy. Soc. Edinburg Sect. A*, **136(3)** (2006), 445–472.
- [3] Ben Yakoub, L. and Louly, A., Derivations of the algebra  $U_q^+(B_2)$ , *Beiträge Algebra and Geom.*, **Vol 50 (2)** (2009), 301–308.
- [4] Benkart, G. and Roby, T., Down-up algebras, *J. Algebra*, **209**(1998), 305–344. Addendum **213**(1999), 378.
- [5] Benkart, G. and Witherspoon, S., Two-parameter quantum groups and Drinfeld doubles, *Algebr. Represent. Theory*, **7** (2004), 261–286.

- [6] Benkart, G. and Witherspoon, S., Representations of two-parameter quantum groups and Schur–Weyl duality, Hopf algebras, Lecture Notes in Pure and Appl. Math., **237**, pp. 65–92, Dekker, New York, 2004.
- [7] Fleury, O., Automorphisms of  $\check{U}_q(\mathfrak{b}^+)$ , *Beiträge Algebra and Geom.*, **Vol 38(2)** (1997), 343–356.
- [8] Launois, S. and Lopes, S.A., Automorphisms and derivations of  $U_q(\mathfrak{sl}_4^+)$ , *J. Pure Appl. Algebra*, **211**(2007), 249–264.
- [9] Osborn J.M. and Passman, D.S., Derivations of skew polynomial rings, *J. Algebra*, **176**(1995), 417–448.
- [10] Tang, X., Ringel-Hall algebras and two-parameter quantized enveloping algebras, *Pacific J. Math.*, **247 no. 1** (2010), 213–240.
- [11] Tang, X., (Hopf) algebra automorphisms of the Hopf algebra  $\check{U}_{r,s}^{\geq 0}(\mathfrak{sl}_3)$ , submitted.

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