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Xin Tang's research paper on two-parameter quantum groups and Ringel-Hall algebras of infinite type

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TWO-PARAMETER QUANTUM GROUPS AND RINGEL-HALL ALGEBRAS OF A_∞–TYPE

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Abstract. In this paper, we study the two-parameter quantum group $U_{r,s}(\mathfrak{sl}_\infty)$ associated to the Lie algebra $\mathfrak{sl}_\infty$ of infinite rank. We shall prove that the two-parameter quantum group $U_{r,s}(\mathfrak{sl}_\infty)$ admits both a Hopf algebra structure and a triangular decomposition. In particular, it can be realized as the Drinfeld double of it's certain Hopf subalgebras. We will also study a two-parameter twisted Ringel-Hall algebra $H_{r,s}(A_\infty)$ associated to the category of all finite dimensional representations of the infinite linear quiver A_∞. In particular, we will establish an iterated skew polynomial presentation of $H_{r,s}(A_\infty)$ and prove that $H_{r,s}(A_\infty)$ is a direct limit of the directed system of the two-parameter Ringel-Hall algebras $H_{r,s}(A_n)$ associated to the finite linear quiver A_n. As a result, we construct a PBW basis for $H_{r,s}(A_\infty)$ and prove that all prime ideals of $H_{r,s}(A_\infty)$ are completely prime. Furthermore, we will establish an algebra isomorphism from $U^+_{r,s}(\mathfrak{sl}_\infty)$ to $H_{r,s}(A_\infty)$, which enable us to obtain the corresponding results for $U^+_{r,s}(\mathfrak{sl}_\infty)$. Finally, via the theory of generic extensions in the category of finite dimensional representations of A_∞, we shall construct several monomial bases and a bar-invariant basis for $U^+_{r,s}(\mathfrak{sl}_\infty)$.

Introduction

As generalizations or variations of the notation of quantum groups [13], several multi-parameter quantum groups have appeared in the literatures [1, 8, 11, 12, 16, 18, 25, 31, 32]. Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra. Let us choose $r, s \in \mathbb{C}^*$ in such a way that r, s are transcendental over $\mathbb{Q}$. The study of the two-parameter quantum group $U_{r,s}(\mathfrak{g})$ has been revitalized in [3, 4, 5, 6, 7] and the references therein. Note that the one-parameter quantum groups associated to Lie algebras $\mathfrak{gl}_\infty, \mathfrak{sl}_\infty$ of infinite ranks [17] have been studied

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in the literatures [10, 19, 22, 23]. Similar to the case of one-parameter quantum groups, one might be interested in the constructions of the corresponding two-parameter quantum groups.

It is the purpose of this paper to study the two-parameter quantum group $U_{r,s}(\mathfrak{sl}_\infty)$, where the Lie algebra $\mathfrak{sl}_\infty$ consists of all infinite trace-zero square matrices with only finitely many non-zero entries. We shall first define such a two-parameter quantum group and then study some of its basic properties. As a matter of fact, we will formulate the two-parameter quantum group $U_{r,s}(\mathfrak{sl}_\infty)$ as a limit version of the two-parameter quantum groups $U_{r,s}(\mathfrak{sl}_{n+1}), n \geq 1$ as studied in [5]. As usual, we will prove that the algebra $U_{r,s}(\mathfrak{sl}_\infty)$ admits a Hopf algebra structure and it is the Drinfeld double of its certain Hopf subalgebras.

To further investigate the structure of the two-parameter quantum group $U_{r,s}(\mathfrak{sl}_\infty)$, we shall study its subalgebra $U_{r,s}^+(\mathfrak{sl}_\infty)$ employing the approach of Ringel-Hall algebras. Note that the Ringel-Hall algebra approach has found many important applications in the study of one-parameter quantum groups [15, 20, 21, 24, 26, 27, 28, 29, 30, 34] and the references therein. To this end, we shall first define a two-parameter twisted Ringel-Hall algebra $H_{r,s}(A_\infty)$ associated to the category of all finite dimensional representations of the infinite linear quiver A_∞. Then we shall prove that the algebra $H_{r,s}(A_\infty)$ can be presented as an iterated skew polynomial ring, and thus construct a PBW basis for $H_{r,s}(A_\infty)$. Furthermore, we shall prove that $H_{r,s}(A_\infty)$ is a direct limit of the two-parameter twisted Ringel-Hall algebras $H_{r,s}(A_n), n \geq 1$ associated to the finite linear quivers $A_n, n \geq 1$ (See [24, 33]). As an application, we prove that all prime ideals of $H_{r,s}(A_\infty)$ are indeed completely prime.

To transfer the information to the algebra $U_{r,s}^+(\mathfrak{sl}_\infty)$, we will establish an algebra isomorphism from $U_{r,s}^+(\mathfrak{sl}_\infty)$ onto $H_{r,s}(A_\infty)$. On the one hand, such an algebra isomorphism provides a generator-relation presentation of the two-parameter Ringel-Hall algebra $H_{r,s}(A_\infty)$, which has been defined over a prescribed basis. On the other hand, via this isomorphism, we can prove that the algebra $U_{r,s}^+(\mathfrak{sl}_\infty)$ can be presented as an iterated skew polynomial ring and it is a direct limit of $U_{r,s}^+(\mathfrak{sl}_{n+1}), n \geq 1$. As a result, we are able to construct a PBW basis for $U_{r,s}^+(\mathfrak{sl}_\infty)$ and prove that all prime ideals of $U_{r,s}^+(\mathfrak{sl}_\infty)$ are completely prime.

To study the Borel subalgebras $U_{r,s}^{\geq 0}(\mathfrak{sl}_\infty)$ (resp. $U_{r,s}^{\leq 0}(\mathfrak{sl}_\infty)$) of the two-parameter quantum group $U_{r,s}(\mathfrak{sl}_\infty)$, we will define a Hopf algebra structure on the extended two-parameter twisted Ringel-Hall algebra $\overline{H_{r,s}(A_\infty)}$ and establish an Hopf algebra isomorphism from the two-parameter quantized Hopf algebra $U_{r,s}^{\geq 0}(\mathfrak{sl}_\infty)$ onto $\overline{H_{r,s}(A_\infty)}$. We will follow the lines in [15, 34]. This result shall provide a realization of the whole two-parameter quantum group $U_{r,s}(\mathfrak{sl}_\infty)$ via the double of two-parameter extended Ringel-Hall algebras.

Note that there exists a $\mathbb{Q}-$algebra automorphism (which will be called the bar-automorphism) on the algebra $U^+_{r,s}(\mathfrak{sl}_\infty)$, which exchanges $r^{\pm 1}$ and $s^{\pm 1}$ and fixes the generators e_i. Using the theory of generic extensions in the category of finite dimensional representations of A_∞, we will construct several monomial bases for the two-parameter quantum groups following the idea used in [9, 24]. As an application, we will also construct a bar-invariant basis for the algebra $U^+_{r,s}(\mathfrak{sl}_\infty)$ following [24].

The paper is organized as follows. In Section 1, we give the definition of $U_{r,s}(\mathfrak{sl}_\infty)$ and study some of it's basic properties. In Section 2, we define and study two-parameter Ringel-Hall algebra $H_{r,s}(A_\infty)$ and establish the algebra isomorphism from $U^+_{r,s}(\mathfrak{sl}_\infty)$ onto $H_{r,s}(A_\infty)$. In Section 3, we define and study the extended two-parameter Ringel-Hall algebra $\overline{H_{r,s}(A_\infty)}$ and establish the Hopf algebra isomorphism from $U^{\geq 0}_{r,s}(\mathfrak{g})$ onto $\overline{H_{r,s}(A_\infty)}$. In Section 4, we use generic extension theory to construct some monomial bases and a bar-invariant basis for $U^+_{r,s}(\mathfrak{sl}_\infty)$.

1. Definition and basic properties of the two-parameter quantum groups $U_{r,s}(\mathfrak{sl}_\infty)$

First of all, let us fix some notation. Let r, s be two-parameters chosen from $\mathbb{C}^*$, such that r, s are transcendental over the field $\mathbb{Q}$ and $r^m s^n = 1$ implies $m = n = 0$. Let us set $\mathcal{Z} = \mathbb{Z}[r^{\pm 1}, s^{\pm 1}]$ and $\mathcal{A} = \mathbb{Q}[r, s]_{(r-1,s-1)}$, which is the localization of $\mathbb{Q}[r, s]$ at the maximal ideal $(r-1, s-1)$.

Let $\mathfrak{sl}_\infty$ denote the infinite dimensional complex Lie algebra which consists of all trace-zero square matrices $(a_{ij})_{i,j\in\mathbb{N}}$ with only finitely many non-zero entries. The one-parameter quantum groups $U_q(\mathfrak{sl}_\infty)$ associated to $\mathfrak{sl}_\infty$ were studied by various people in the references [10, 19, 22, 23]. Following a similar idea in [5, 32], we will introduce a class of two-parameter quantum groups $U_{r,s}(\mathfrak{sl}_\infty)$ associated to the Lie algebra $\mathfrak{sl}_\infty$.

It is well known that one can also define roots for the Lie algebra $\mathfrak{sl}_\infty$ as in the finite dimensional case of $\mathfrak{g} = \mathfrak{sl}_{n+1}$. In particular, all the simple roots of $\mathfrak{sl}_\infty$ can be denoted as $\alpha_i, i \in I = \mathbb{N}$. Accordingly, all the positive roots of $\mathfrak{sl}_\infty$ are exactly given as $\alpha_{ij} := \sum_{k=i}^{j} \alpha_k$ for $i \leq j \in \mathbb{N}$.

Let $C=(c_{ij})_{i,j\in\mathbb{N}}$ denote the infinite Cartan matrix corresponding to the Lie algebra $\mathfrak{sl}_\infty$. Then, we have the following

$$c_{ii}=2,\ c_{ij}=-1\,\text{for}\,|i-j|=1,\ c_{ij}=0\,\text{for}\,|i-j|>1.$$

Let $\mathbb{Q}(r,s)$ denote the function field in two variables r,s over the field $\mathbb{Q}$ of all rational numbers. As mentioned early, we may also choose r,s to be complex numbers which are transcendental over $\mathbb{Q}$ such that $r^2\neq s^2$. In the future, more restrictions may be put on the parameters if needed. Let $\mathcal{Q}$ denote the root lattice generated by $\alpha_i, i\in\mathbb{N}$. Then we can define a bilinear form $\langle -,-\rangle$ on the root lattice $\mathcal{Q}\cong\mathbb{Z}^{\oplus\mathbb{N}}$ as follows

$$\langle i,j\rangle:\ =\langle\alpha_i,\alpha_j\rangle=\begin{cases} a_{ij}, & \text{if } i<j,\\ 1, & \text{if } i=j,\\ 0, & \text{if } i>j.\end{cases}$$

The above defined bilinear form is a generalization of the Ringel-Euler form in the finite case of $\mathfrak{g}=sl_{n+1}$. In terms of the above defined bilinear form $\langle -,-\rangle$, one can give the following definition of the two-parameter quantum groups $U_{r,s}(\mathfrak{sl}_\infty)$ associated to the Lie algebra $\mathfrak{sl}_\infty$.

Definition 1.1. The two-parameter quantum group $U_{r,s}(\mathfrak{sl}_\infty)$ is defined to be the $\mathbb{Q}(r,s)-$algebra generated by $e_i, f_i, w_i^{\pm1}, w_i'^{\pm1}, i\in\mathbb{N}$ subject to the following relations

$$\begin{aligned}
w_i^{\pm1}w_j^{\pm1}=w_j^{\pm1}w_i^{\pm1},\quad w_i'^{\pm1}w_j'^{\pm1}&=w_j'^{\pm1}w_i'^{\pm1},\\
w_i^{\pm1}w_j'^{\pm1}=w_j'^{\pm1}w_i^{\pm1},\quad w_i^{\pm1}w_i^{\mp1}&=1=w_i'^{\pm1}w_i'^{\mp1},\\
w_ie_j=r^{\langle j,i\rangle}s^{-<i,j>}e_jw_i,\quad w_i'e_j&=r^{-\langle i,j\rangle}s^{\langle j,i\rangle}e_jw_i',\\
w_if_j=r^{-\langle j,i\rangle}s^{\langle i,j\rangle}f_je_i,\quad w_i'f_j&=r^{\langle i,j\rangle}s^{-\langle j,i\rangle}f_jw_i',\\
e_if_j-f_je_i&=\delta_{i,j}\frac{w_i-w_i'}{r_i-s_i},\\
e_ie_j-e_je_i=f_if_j-f_jf_i&=0\,\text{for}\,|i-j|>1,\\
e_i^2e_{i+1}-(r+s)e_ie_{i+1}e_i+rse_{i+1}e_i^2&=0,\\
e_ie_{i+1}^2-(r+s)e_{i+1}e_ie_{i+1}+rse_{i+1}^2e_i&=0,\\
f_i^2f_{i+1}-(r^{-1}+s^{-1})f_if_{i+1}f_i+(rs)^{-1}f_{i+1}f_i^2&=0,\\
f_if_{i+1}^2-(r^{-1}+s^{-1})f_{i+1}f_if_{i+1}+(rs)^{-1}f_{i+1}^2f_i&=0.
\end{aligned}$$

First of all, we have the following obvious proposition concerning a Hopf algebra structure of the algebra $U_{r,s}(\mathfrak{sl}_\infty)$.

Proposition 1.1. *The algebra $U_{r,s}(\mathfrak{g})$ is a Hopf algebra with the comultiplication, counit and antipode defined as follows*

$$\begin{aligned}\Delta(w_i^{\pm 1}) = w_i^{\pm 1} \otimes w_i^{\pm 1}, \quad \Delta(w_i'^{\pm 1}) = w_i'^{\pm 1} \otimes w_i'^{\pm 1},\\ \Delta(e_i) = e_i \otimes 1 + w_i \otimes e_i, \quad \Delta(f_i) = 1 \otimes f_i + f_i \otimes w_i',\\ \epsilon(w_i^{\pm 1}) = \epsilon(w_i'^{\pm 1}) = 1, \quad \epsilon(e_i) = \epsilon(f_i) = 0,\\ S(w_i^{\pm 1}) = w_i^{\mp 1}, \quad S(w_i'^{\pm 1}) = w_i'^{\mp 1},\\ S(e_i) = -w_i^{-1} e_i, \quad S(f_i) = -f_i w_i'^{-1}.\end{aligned}$$

Proof: The proof is reduced to the finite case where $\mathfrak{g} = \mathfrak{s}l_{n+1}$, whose proof can be found in [5]. And we will not repeat the details here. □

Let $U_{r,s}^+(\mathfrak{s}l_\infty)$ (resp. $U_{r,s}^-(\mathfrak{s}l_\infty)$) denote the subalgebra of $U_{r,s}(\mathfrak{s}l_\infty)$ generated by $e_i, i \in \mathbb{N}$ (resp. by $f_i, i \in \mathbb{N}$). Let $U_{r,s}^0(\mathfrak{s}l_\infty)$ denote the subalgebra of $U_{r,s}(\mathfrak{s}l_\infty)$ generated by $w_i^{\pm 1}, w_i'^{\pm 1}, i \in \mathbb{N}$. Then we shall have the following triangular decomposition of $U_{r,s}(\mathfrak{s}l_\infty)$.

Proposition 1.2. *The algebra $U_{r,s}(\mathfrak{s}l_\infty)$ has a triangular decomposition*

$$U_{r,s}(\mathfrak{s}l_\infty) \cong U_{r,s}^-(\mathfrak{s}l_\infty) \otimes U_{r,s}^0(\mathfrak{s}l_\infty) \otimes U_{r,s}^+(\mathfrak{s}l_\infty).$$

Proof: Once again, we can repeat the proof used in the case of $U_{r,s}(\mathfrak{s}l_{n+1})$. We refer the reader to [5] for more details. □

Let us denote by $\mathbb{Z}^{\oplus\mathbb{N}}$ the free abelian group of rank $|\mathbb{N}|$ with a basis denoted by $z_1, z_2, \cdots, z_n, \cdots$. Given any element $\mathbf{a} \in \mathbb{Z}^{\oplus\mathbb{N}}$, say $\mathbf{a} = \sum a_i z_i$, we set $|\mathbf{a}| = \sum a_i$. Note that algebra $U_{r,s}^+(\mathfrak{sl}_\infty)$ (resp. $U_{r,s}^-(\mathfrak{s}l_\infty)$) is a $\mathbb{Z}^{\oplus\mathbb{N}}-$graded algebra by assigning to the generator e_i (resp. f_i) the degree z_i. Given $\mathbf{a} \in \mathbb{Z}^{\oplus\mathbb{N}}$, we denote by $U_{r,s}^{\pm}(\mathfrak{s}l_\infty)_{\mathbf{a}}$ the set of homogeneous elements of degree $\mathbf{a}$ in $U_{r,s}^{\pm}(\mathfrak{s}l_\infty)$.

Proposition 1.3. *We have the following decomposition*

$$U_{r,s}^+(\mathfrak{sl}_\infty) = \bigoplus_{\mathbf{a}} U_{r,s}^+(\mathfrak{sl}_\infty)_{\mathbf{a}}, \quad U_{r,s}^-(\mathfrak{sl}_\infty) = \bigoplus_{\mathbf{a}} U_{r,s}^-(\mathfrak{sl}_\infty)_{\mathbf{a}}.$$

□

Let us define $U_{v,v^{-1}}(\mathfrak{s}l_\infty)$ to be the specialization of $U_{r,s}(\mathfrak{s}l_\infty)$ for $r = v = s^{-1}$. Then we shall have the following similar result as [5], whose proof is exactly the same as the one in [5].

Proposition 1.4. *Assume there exists an isomorphism of Hopf algebras*

$$\phi\colon U_{r,s}(\mathfrak{s}l_\infty) \longrightarrow U_{v,v^{-1}}(\mathfrak{s}l_\infty)$$

for some v. Then $r = v$ and $s = v^{-1}$.

□

1.1. **A Drinfeld double realization of $U_{r,s}(\mathfrak{sl}_\infty)$.** In this subsection, we show that the two-parameter quantum group $U_{r,s}(\mathfrak{sl}_\infty)$ can be realized as the Drinfeld double of its certain Hopf subalgebras. To proceed, we need to recall a couple of standard definitions for the Hopf pairing and the Drinfeld double of Hopf algebras. For more details about these concepts, we refer the reader to the references [5, 12].

Definition 1.2. A Hopf pairing of two Hopf algebras H' and H is a bilinear form $(,)\colon H' \times H \longrightarrow \mathbb{K}$ such that

(1) $(1, h) = \epsilon_H(h)$,

(2) $(h', 1) = \epsilon_{H'}(h')$,

(3) $(h', hk) = (\Delta_{H'}(h'), h \otimes k) = \sum (h'_{(1)}, h)(h'_{(2)}, k)$,

(4) $(h'k', h) = (h' \otimes k', \Delta_{\prime}(h)) = \sum (h', h_{(1)})(k', h_{(2)})$,

for all $h, k \in H'$,$h', k' \in H'$, where $\epsilon_H, \epsilon_{H'}$ denote the counits of H, H' respectively, and $\Delta_H, \Delta_{H'}$ denote their comultiplications.

It is obvious that

$$(S_{H'}(h'), h) = (h', S_H(h))$$

for all $h \in H$ and $h' \in H'$, where $S_{H'}$ and S_H denote the respective antipodes of H and H'.

Let $U^{\geq 0}_{r,s}(\mathfrak{sl}_\infty)$ (resp. $U^{\leq 0}_{r,s}(\mathfrak{sl}_\infty)$) be the Hopf subalgebra of $U_{r,s}(\mathfrak{sl}_\infty)$ generated by $e_i, w_i^{\pm 1}$ (resp. $f_i, w_i'^{\pm 1}$). Assume that $B = U^{\geq 0}_{r,s}(\mathfrak{sl}_\infty)$ and $(B')^{coop}$ is the Hopf algebra generated by $f_j, (w_j')^{\pm 1}$ with the opposite coproduct to $U^{\leq 0}(\mathfrak{s}l_\infty)$. Using the same proof in the case of $\mathfrak{s}l_{n+1}$ [5], we shall have the following result

Lemma 1.1. *There exists a unique Hopf pairing B and B' such that*

$$(f_i, e_j) = \frac{\delta_{i,j}}{s - r}$$
$$(w_i', w_j) = r^{<e_i,e_j>} s^{-<e_j,e_i>},$$

and the pairing takes the zero value on all other pairs of generators. Moreover, we have $(S(a), S(b)) = (a, b)$ for $a \in B', b \in B$.

□

Therefore, we have the following similar result as in [5].

Theorem 1.1. *$U_{r,s}(\mathfrak{sl}_\infty)$ can be realized as a Drinfeld double of Hopf subalgebras $B = U^{\geq 0}_{r,s}(\mathfrak{sl}_\infty)$ and $(B')^{coop} = U^{\leq 0}_{r,s}(\mathfrak{sl}_\infty)$, that is,*

$$U_{r,s}(\mathfrak{sl}_\infty) \cong D(B, (B')^{coop}).$$

Proof: First of all, let us define a linear map: $\phi\colon D(B,(B')^{coop}) \longrightarrow U_{r,s}(\mathfrak{s}l_\infty)$ as follows

$$\phi(\hat{\omega}_i^{\pm 1}) = \omega_i^{\pm 1}, \quad \phi((\hat{\omega}_i')^{\pm 1}) = (\omega_i')^{\pm 1}$$

$$\phi(\hat{e}_i) = e_i, \quad \phi_i(\hat{f}_i) = f_i.$$

We need to show that this mapping is a Hopf algebra automorphism. Obviously, we can still employ the proof used in [5] for the finite case $g = \mathfrak{s}l_{n+1}$ and we will not repeat the detail here. □

1.2. **An integral form of the two-parameter quantum group $U_{r,s}(\mathfrak{s}l_\infty)$.** In addition, we also consider an integral form of the two-parameter quantum group $U_{r,s}(\mathfrak{s}l_\infty)$ and its subalgebras following [20]. Recall that we have set $\mathcal{Z} = \mathbb{Z}[r^{\pm 1}, s^{\pm 1}]$. For any $l \geq 1$, let us set the following

$$[l] = \frac{r^l - s^l}{r - s},\ [l]^! = [1][2]\cdots[l].$$

Let us define $e_i^{(l)} = \frac{e_i^l}{[l]^!}, f_i^{(l)} = \frac{f_i^l}{[l]^!}$. We define a $\mathcal{Z}-$subalgebra $U_{r,s}(\mathfrak{s}l_\infty)_{\mathcal{Z}}$ of $U_{r,s}(\mathfrak{s}l_\infty)$ which is generated by the elements $e_i^{(l)}, f_i^{(l)}, w_i^{\pm 1}, w_i'^{\pm 1}$ for $i \in I$. Similarly, we can define the integral form of $U_{r,s}^+(\mathfrak{s}l_\infty)$ and $U_{r,s}^-(\mathfrak{s}l_\infty)$. It is easy to see that we have the following

$$U_{r,s}(\mathfrak{s}l_\infty) \cong U_{r,s}(\mathfrak{s}l_\infty)_{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathbb{Q}(r,s)$$

and

$$U_{r,s}^{\pm}(\mathfrak{s}l_\infty) \cong U_{r,s}^{\pm}(\mathfrak{s}l_\infty)_{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathbb{Q}(r,s).$$

In particular, $U_{r,s}(\mathfrak{s}l_\infty)$ (resp. $U_{r,s}^{\pm}(\mathfrak{s}l_\infty)$) is a free $\mathcal{Z}-$algebra.

2. Two-parameter Ringel-Hall algebras $H_{r,s}(A_\infty)$

To better understand the structure of the two-parameter quantum group $U_{r,s}(\mathfrak{sl}_\infty)$, it is helpful to study its subalgebras $U_{r,s}^+(\mathfrak{sl}_\infty)$ and (Hopf) subalgebra $U_{r,s}^{\geq 0}(\mathfrak{sl}_\infty)$. To this end, we shall study these algebras in terms of two-parameter Ringel-Hall algebras associated to the infinite linear quiver. In this section, we will define and study a two-parameter Ringel-Hall algebra $H_{r,s}(A_\infty)$ associated to the category of finite dimensional representations of the infinite quiver

$$A_\infty\colon \overset{1}{\bullet}\longrightarrow\overset{2}{\bullet}\longrightarrow\overset{3}{\bullet}\cdots\overset{n-1}{\bullet}\longrightarrow\overset{n}{\bullet}\longrightarrow\cdots.$$

For $n \geq 1$, let A_n denote the finite quiver

$$A_n\colon \overset{1}{\bullet}\longrightarrow\overset{2}{\bullet}\longrightarrow\overset{3}{\bullet}\cdots\overset{n-1}{\bullet}\longrightarrow\overset{n}{\bullet}$$

with n vertices. Let us fix k to be a finite field and let Λ_n denote the path algebra of the finite linear quiver A_n over k. Then Λ_n is a finite dimensional hereditary algebra of finite-representation type. Note that the category of finite dimensional representations of the quiver A_n is equivalent to the category of finite dimensional Λ_n-modules. We will denote this category by A_n-mod. Let us set $q = |k|$ the cardinality of k, and choose v to be a number such that $v^2 = q$. We know that Λ_n is finitary in the sense that the cardinality of the extension group $Ext^1(S, S')$ is finite for any two simple Λ_n-modules S, S'.

Let us denote by $A_\infty-$mod, the category of all finite dimensional representations of the quiver A_∞. Note that the category A_∞-mod has been investigated by Hou and Ye in [14], where they have explicitly described all finite dimensional indecomposable representations of A_∞ and studied the one-parameter non-twisted generic Ringel-Hall algebra $H_q(A_\infty)$. Let S_i be the simple representation associated to the vertex i of the quiver A_∞ and let M_{ij} denote the indecomposable representation of A_∞ with a top S_i and length $j - i + 1$. It is easy to see that there is a one-one to correspondence between the set of isoclasses of finite dimensional indecomposable representations M_{ij} of the quiver A_∞ and the set of positive roots α_{ij} for the Lie algebra $\mathfrak{sl}_\infty$.

Concerning the relationship between the categories A_n-mod and $A_\infty-$mod, we now recall the following result from [14].

Theorem 2.1. *(***Theorem 1.1** *in* [14]*) The category A_n-mod can be regarded as a fully faithful and extension closed subcategory of $A_\infty-$mod and A_m-mod for $m \geq n$.*

□

Based on the above theorem, we know that the extension group between any two finite dimensional representations M, N of A_∞ can be calculated via regarding M, N as the representations of a certain finite quiver A_m. Therefore, the number of extensions between M, N is still depicted by the evaluation of the Hall polynomial at q, the cardinality of the base field. Recall that the two-parameter Ringel-Hall algebra $H_{r,s}(A_n), n \geq 1$ associated to the category A_n-mod has been studied in [24, 33]. In particular, one knows that $H_{r,s}(A_n)$ can be presented as an iterated skew polynomial ring and its prime ideals are completely prime. A PBW basis has also been constructed for $H_{r,s}(A_n)$ in [33] as well. In rest of this section, we are going to use a two-parameter twisted version of the Ringel-Hall algebra $H_q(A_\infty)$ to study the algebra $U_{r,s}^+(\mathfrak{sl}_\infty)$. Note that this approach is plausible because of the existence of Hall polynomials in the category $A_\infty-$mod. Indeed, we will be looking at a limit version $H_{r,s}(A_\infty)$ of the two-parameter Ringel-Hall algebras $H_{r,s}(A_n), n \geq 1$.

2.1. **Two-parameter Ringel-Hall algebra $H_{r,s}(A_\infty)$.** We will denote by $\mathcal{P}$ the set of isomorphism classes of finite dimensional representations of the infinite quiver A_∞. Let us define the subset

$$\mathcal{P}_1 = \mathcal{P} - 0$$

where 0 denotes the subset of $\mathcal{P}$ consisting of the only isomorphism class of the zero representation. For any $\alpha \in \mathcal{P}$, we choose a representation u_α corresponding to α. We denote by a_α the order of the automorphism group $Aut(u_\alpha)$. It is easy to see that the number a_α is independent of the choices of the representatives u_α for any $\alpha \in \mathcal{P}$.

For any given three representatives $u_\alpha, u_\beta, u_\gamma$ of the elements $\alpha, \beta, \gamma \in \mathcal{P}$ respectively, we denote by $g^\gamma_{\alpha\beta}$ the number of submodules N of u_γ satisfying the conditions: $N \cong u_\beta$ and $u_\gamma/N \cong u_\alpha$.

Note that it does not make sense to define $Ext^1(M, N)$ for any two given representations M, N of the infinite quiver A_∞. Let us denote by $\hat{E}_{A_\infty}(M, N)$ the set of all short exact sequences $0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$. We say two such short exact sequences $0 \longrightarrow N \longrightarrow E_1 \longrightarrow M \longrightarrow 0$ and $0 \longrightarrow N \longrightarrow E_2 \longrightarrow M \longrightarrow 0$ are equivalent if there exists a homomorphism ϕ: $E_1 \longrightarrow E_2$ making the diagram commute. We denote by $E_{A_\infty}(M, N)$ the set of all equivalence classes of $\hat{E}_{A_\infty}$ with respect to this equivalence relation. For any given $M, N \in A_\infty - mod$, according to **Theorem 1.2** in [14], we can choose some $m \geq 1$ such that there exists a bijection between $E_{A_\infty}(M, N)$ and $Ext^1_{A_m - mod}(M, N)$. If no confusion arises, we will still write $E_{A_\infty}(M, N)$ as $Ext^1(M, N)$.

For any given $M, N \in A_\infty - mod$, we define the following notation

$$\langle M, N\rangle = dim_k Hom(M, N) - dim_k Ext^1(M, N).$$

Once the representations M, N are chosen, we can always restrict to a subcategory $A_n - mod$. Since the algebra Λ_n is hereditary for any $n \in \mathbb{N}$, it is easy to see that for any representations $M, N \in A_n - mod$, the value of $\langle M, N\rangle$ solely depends on the dimension vectors $\underline{dim}M$, $\underline{dim}N$ of the A_n−modules M and N.

Now for any given elements $\alpha, \beta \in \mathcal{P}$, we can define the following notation

$$\langle \alpha, \beta\rangle = \langle u_\alpha, u_\beta\rangle$$

where u_α, u_β are any chosen representatives of α, β respectively. It is easy to see that $\langle -, -\rangle$ is a bilinear form.

It is well known that in the category A_n−mod, there exists a symmetry between the objects of A_n−mod. This symmetry is described by Green's formula [15]. In fact, one can also prove that Green's formula holds for the objects in the category A_∞−mod. Namely, we have the following result.

Theorem 2.2. *Let $\alpha, \beta, \alpha', \beta' \in \mathcal{P}$, then we have*

$$a_\alpha a_\beta a_{\alpha'} a_{\beta'} \sum_{\lambda \in \mathcal{P}} g^\lambda_{\alpha,\beta} g^\lambda_{\alpha'\beta'} a_\lambda^{-1} = \sum_{\rho,\sigma,\sigma',\tau \in \mathcal{P}} \frac{|Ext^1(u_\rho, u_\tau)|}{|Hom(u_\rho, u_\tau)|} \quad g^\alpha_{\rho\sigma} g^{\alpha'}_{\rho\sigma'} g^\beta_{\sigma'\tau} g^{\beta'}_{\sigma\tau}$$
$$a_\rho a_\sigma a_{\sigma'} a_{\tau'}.$$

Proof: Since all representations involved in the formula are finite dimensional representations of A_∞, we can choose some positive integer m such that $\alpha, \beta, \alpha', \beta'$ and λ can actually be regarded as objects in the subcategory A_m-mod instead. Note that Green's formula holds within the subcategory A_m-mod. Since the category A_m-mod is a fully faithful and extension closed subcategory of $A_\infty-$mod, we know that Green's formula holds in $A_\infty-$mod. □

Let $H_{r,s}(A_n)$ denote the two-parameter Ringel-Hall algebra associated to the category A_n-mod as defined in [24]. In [24], Reineke has proved that the two-parameter Ringel-Hall algebra $H_{r,s}(A_n)$ is isomorphic to the algebra $U^+_{r,s}(\mathfrak{s}ln+1)$. In the rest of this section, we will show that a limit version of this statement is still true.

Note that there exist Hall polynomials $F^L_{M,N}(x)$ for $M, N, L \in A_n-mod$ such that $g^L_{M,N} = F^L_{M,N}(q)$, where q is the cardinality of the base field k. For the existence and calculation of Hall polynomials in A_n-mod, we refer the reader to the references [27, 28]. Since each A_n-mod is a fully faithful and extension closed subcategory of $A_\infty-$mod, the Hall polynomials exists for objects in $A_\infty-$mod, which leads to the definition of two-parameter Ringel-Hall algebra $H_{r,s}(A_\infty)$ below.

Now let us define $H_{r,s}(A_\infty)$ to be the free $\mathbb{Q}(r,s)-$module generated by the set $\{u_\alpha \mid \alpha \in \mathcal{P}\}$. Moreover, we define a multiplication on the free $\mathbb{Q}(r,s)-$module $H_{r,s}(A_\infty)$ as follows

$$u_\alpha u_\beta = \sum_{\lambda \in \mathcal{P}} s^{-\langle \alpha,\beta \rangle} F^{u_\lambda}_{u_\alpha u_\beta}(rs^{-1}) u_\lambda, \quad \text{for any}\, \alpha, \beta \in \mathcal{P}.$$

It is easy to see that we have the following result.

Theorem 2.3. *The free $\mathbb{Q}(r,s)-$module $H_{r,s}(A_\infty)$ is an associative $\mathbb{Q}(r,s)-$algebra under the above defined multiplication. In particular, the algebra $H_{r,s}(A_n)$ can be regarded as a subalgebra of $H_{r,s}(A_\infty)$ and $H_{r,s}(A_m)$ for $m \geq n$. In particular, we have*

$$H_{r,s}(A_\infty) = \lim_{n \mapsto \infty} H_{r,s}(A_n).$$

Proof: It is straightforward to verify that $H_{r,s}(A_\infty)$ is an associative algebra under the above defined multiplication. Once again, we can reduce the proof to the finite case thanks to **Theorem 1.1** in [14]. Since each A_n-mod can be regarded as a fully faithful and extension closed subcategory of $A_\infty-$mod and A_m-mod when $m \geq n$, the algebra $H_{r,s}(A_n)$ can be regarded as a subgroup of the algebras $H_{r,s}(A_\infty)$ and $H_{r,s}(A_m)$. Furthermore, one notices that the multiplication of $H_{r,s}(A_n)$ is the restriction of the multiplications of $H_{r,s}(A_\infty)$ and $H_{r,s}(A_m)$. Therefore, the algebra $H_{r,s}(A_n)$ can be regarded as a subalgebra of $H_{r,s}(A_\infty)$ and $H_{r,s}(A_m)$ for $m \geq n$ as desired. Furthermore, each element of $H_{r,s}(A_\infty)$ can be regarded as an element of a certain subalgebra $H_{r,s}(A_m)$. Thus we shall have $H_{r,s}(A_\infty) = \lim_{n \mapsto \infty} H_{r,s}(A_n)$ as desired. □

2.2. **Basic properties of $H_{r,s}(A_\infty)$.** Since the category $A_\infty-$mod can be regarded the direct limit of its fully faithful and extension closed subcategories A_n-mod with $n \geq 1$, any two objects $M, N \in A_\infty - mod$ can be regarded as objects in a certain subcategory A_m-mod. Thus the extension between any such two objects can be handled in this subcategory A_n-mod as well. As a result, it is no surprise that the algebra $H_{r,s}(A_\infty)$ shares many similar ring-theoretic properties with its subalgebras $H_{r,s}(A_n)$. In this subsection, we will establish some similar results for $H_{r,s}(A_\infty)$ without giving detailed proofs. The reader shall be reminded that all the proofs can be reconstructed the same way as in the case of a certain subalgebra $H_{r,s}(A_m)$. And we refer the curious reader to [33] for the details.

First of all, let us fix more notations. For any given $\alpha \in \mathcal{P}$, we will choose an element $u_\alpha \in H_{r,s}(\Lambda)$. We denote by $\epsilon(\alpha)$ the $k-$dimension of the endomorphism ring of the representative u_α associated to α. For any given finite dimensional representation M of the infinite quiver A_∞, we will denote the isomorphism class of M by $[M]$ and the dimension vector of M by $\underline{dim}M$, which is an element of the Grothendieck group $K_0(A_\infty)$ of the category $A_\infty-$mod.

Recall that there is a one-to-one correspondence between the set of all positive roots for the Lie algebra $\mathfrak{sl}_\infty$ and the set of isoclasses of finite dimensional indecomposable representations of A_∞. Let $\mathbf{a} \in \Phi^+$ be any positive root, we shall denote by $M(\mathbf{a})$ the indecomposable representation corresponding to $\mathbf{a}$. For any given map $\alpha\colon \Phi^+ \longrightarrow \mathbb{N}_0$ with finite support, let us set the following

$$M(\alpha) = M_\Lambda(\alpha) = \bigoplus_{\mathbf{a} \in \mathbf{\Phi}^+} \alpha(\mathbf{a})\mathbf{M}(\mathbf{a}).$$

Then it is easy to see there is a one-to-one correspondence between the set $\mathcal{P}$ of isomorphism classes of all finite dimensional representations of the infinite quiver A_∞ and the set of all maps $\alpha\colon \Phi^+ \longrightarrow \mathbb{N}_0$ with finite supports. From now on, we will not distinguish an element $\alpha \in \mathcal{P}$

from the corresponding map associated to α, and we may denote both of them by α if no confusion arises.

For any given $\alpha \in \mathcal{P}$, let us set $\mathbf{dim}\alpha = \sum_{\mathbf{a}\in\mathbf{\Phi}^+} \alpha(\mathbf{a})\mathbf{a}$. Then we shall have following

$$\underline{dim}M(\alpha) = \mathbf{dim}\alpha.$$

For any given $\alpha \in \mathcal{P}$, we will denote by $dim(\alpha) = dim(u_\alpha)$ the dimension of the representation u_α as a $k-$vector space. Furthermore, let us set

$$\langle u_\alpha \rangle = s^{dim(u_\alpha)-\epsilon(\alpha)} u_\alpha.$$

For conveniences, we may sometimes simply denote the element u_α by α for any $\alpha \in \mathcal{P}$, and denote $F^{u_\lambda}_{u_\alpha u_\beta}(rs^{-1})$ by $g^\lambda_{\alpha\beta}$, if no confusion arises. In the rest of this subsection, we will carry out all the computations in terms of α. It is obvious that the set $\{\langle\alpha\rangle \mid \alpha \in \mathcal{P}\}$ is also a $\mathbb{Q}(r,s)-$basis for the algebra $H_{r,s}(A_\infty)$. Note that we have $\langle\alpha_i\rangle = \alpha_i$ for any given element $\alpha_i \in \mathcal{P}$ corresponding to the simple root $\alpha_i, i \geq 1$. As a result, we can rewrite the multiplication of $H_{r,s}(A_\infty)$ in terms of this new basis as follows

$$\langle\alpha\rangle\langle\beta\rangle = s^{-\epsilon(\alpha)-\epsilon(\beta)-\langle \mathbf{dim}\alpha, \mathbf{dim}\beta\rangle} \sum_{\lambda\in\mathcal{P}} s^{\epsilon(\lambda)} g^\lambda_{\alpha\beta} \langle\lambda\rangle$$

for any $\alpha, \beta \in \mathcal{P}$.

In addition, let us denote by

$$e(\alpha,\beta) = dim_k Hom_{A_\infty - mod}(M(\alpha), M(\beta))$$

and

$$\zeta(\alpha,\beta) = dim_k Ext^1_{A_\infty - mod}(M(\alpha), M(\beta)).$$

Recall that Hou and Ye have given an explicit total ordering on the set of all isoclasses of finite dimensional indecomposable representations of the infinite linear quiver A_∞ and used it to construct a PBW base for the generic one-parameter Ringel-Hall algebra $H_q(A_\infty)$. Following [14], we will order all the positive roots as follows:

$$\mathbf{a}_{11} < \mathbf{a}_{12} < \cdots < \mathbf{a}_{22} < \mathbf{a}_{23} < \cdots .$$

Obviously, we can see that $Hom(M(\mathbf{a}_{ij}), M(\mathbf{a}_{kl})) \neq 0$ implies $\mathbf{a}_{ij} > \mathbf{a}_{kl}$, where $M(\mathbf{a}_{ij}), M(\mathbf{a}_{kl})$ are the indecomposable representations corresponding to the positive roots $\mathbf{a}_{ij}, \mathbf{a}_{kl}$ respectively. For more details about the ordering, we refer the reader to [14, 27]. We should mention that we may write the positive roots as $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \cdots$ instead.

First of all, we have the following proposition.

Proposition 2.1. *Let* $\alpha_1, \cdots, \alpha_t \in \mathcal{P}$ *such that for* $i < j$*, we have both* $\epsilon(\alpha_j, \alpha_i) = 0$ *and* $\zeta(\alpha_i, \alpha_j) = 0$*. Then*

$$\langle \bigoplus_{i=1}^{t} \alpha_i \rangle = \langle \alpha_1 \rangle \cdots \langle \alpha_t \rangle.$$

□

Theorem 2.4. *Let* $\alpha, \beta \in \mathcal{P}$ *such that* $e(\beta, \alpha) = 0, \zeta(\alpha, \beta) = 0$*. Then we have the following*

$$\langle \beta \rangle \langle \alpha \rangle = r^{\langle \alpha, \beta \rangle} s^{-\langle \beta, \alpha \rangle} \langle \alpha \rangle \langle \beta \rangle + \sum_{\gamma \in J(\alpha, \beta)} c_\gamma \langle \gamma \rangle$$

with coefficients c_γ *in* $\mathbb{Z}[r^{\pm 1}, s^{\pm 1}]$ *and* $J(\alpha, \beta)$ *is the set of all elements* $\lambda \in \mathcal{P}$ *which are different from* $\alpha \oplus \beta$ *and* $g^{\lambda}_{\alpha\beta} \neq 0$*.*

□

Proposition 2.2. *For any given* $\alpha \in \mathcal{P}$*, we have*

$$\langle \alpha \rangle = \langle \alpha(\mathbf{a}_1)\mathbf{a}_1 \rangle \cdots \langle \alpha(\mathbf{a}_m)\mathbf{a}_m \rangle.$$

□

Now let us consider the divided powers of $\langle \mathbf{a} \rangle$ by setting

$$\langle \mathbf{a} \rangle^{(t)} = \frac{1}{[t]^!_{\epsilon(\mathbf{a})}} \langle \mathbf{a} \rangle^t$$

where $[t]^!_{\epsilon(\mathbf{a})} = \prod_{i=1}^{t} \frac{r^{i\epsilon(\mathbf{a})} - s^{i\epsilon(\mathbf{a})}}{r^{\epsilon(\mathbf{a})} - s^{\epsilon(a)}}$.

Then we have the following lemma.

Lemma 2.1. *Let* $\mathbf{a}$ *be a positive root and* $t \geq 0$ *be an integer. Then we have the following*

$$\langle t\mathbf{a} \rangle = \langle \mathbf{a} \rangle^{(t)}.$$

□

For each positive root $\mathbf{a}_i$, let us define the following symbol

$$X_i = \langle \mathbf{a}_i \rangle.$$

Then we have the following proposition:

Proposition 2.3. *Let* $\alpha \in \mathcal{P}$ *and regard* α *as a map* $\alpha \colon \Phi^+ \longrightarrow \mathbb{N}_0$ *with finite support. Let us set* $\alpha(i) = \alpha(\mathbf{a}_i)$*, then we have the following*

$$\langle \alpha \rangle = X_1^{(\alpha(1))} \cdots X_m^{(\alpha(m))} = (\prod_{i=1}^{m} \frac{1}{[\alpha(i)]^!_{\epsilon(\mathbf{a}_i)}}) X_1^{\alpha(1)} \cdots X_m^{\alpha(m)}.$$

□

Theorem 2.5. *The monomials $X_1^{\alpha(1)} \cdots X_m^{\alpha(m)}$ with $\alpha(1), \cdots, \alpha(m) \in \mathbb{N}_0$ form a $\mathbb{Q}(r,s)-$basis of $H_{r,s}(\Lambda)$; and for $i < j$, we have*

$$\begin{aligned} X_j X_i &= r^{\langle \underline{dim} X_i, \underline{dim} X_j \rangle} s^{-\langle \underline{dim} X_j, \underline{dim} X_i \rangle} X_i X_j \\ &+ \sum_{I(i,j)} c(a_{i+1}, \cdots, a_{j-1}) X_{i+1}^{a_{i+1}} \cdots X_{j-1}^{a_{j-1}} \end{aligned}$$

with coefficients $c(a_{i+1}, \cdots, a_{j-1})$ in $\mathbb{Q}(r,s)$. Here the index set $I(i,j)$ is the set of sequences $(a_{i+1}, \cdots a_{j-1})$ of natural numbers such that $\sum_{t=i+1}^{j-1} a_t \mathbf{a_t} = \mathbf{a_i} + \mathbf{a_j}$.

□

Now we define some algebra automorphisms and skew derivations on $H_{r,s}(A_\infty)$. For any $d \in \mathbb{Z}^{\oplus \mathbb{N}}$, we define an algebra automorphism l_d of $H_{r,s}(A_\infty)$ as follows

$$l_d(w) = r^{<\underline{dim} w, d>} s^{-<d, \underline{dim} w>} w$$

where w is any homogeneous element of $H_{r,s}(A_\infty)$.

Let H_j denote the $\mathbb{Q}(r,s)-$subalgebra of $H_{r,s}(A_\infty)$ generated by the generators $X_1, \cdots, X_j$. Thus we have $H_0 = \mathbb{Q}(r,s)$ and for any $0 \le j \le m$, we have following

$$H_j = H_{j-1}[X_j, l_j, \delta_j]$$

with the automorphism l_j and the l_j-derivation δ_j of H_{j-1}. Note that the automorphism l_j can be explicitly defined as follows

$$l_j(X_i) = r^{\langle \underline{dim} X_i, \underline{dim} X_j \rangle} s^{-\langle \underline{dim} X_j, \underline{dim} X_i \rangle} X_i$$

for $i < j$. And the skew derivation δ_j can be defined as follows:

$$\delta_j(X_i) = X_j X_i - l_j(X_i) X_j = \sum_{I(i,j)} c(a_{i+1}, \cdots, a_{j-1}) X_{i+1}^{a_{i+1}} \cdots X_{j_1}^{a_{j-1}}.$$

It is easy to check that we have the following result.

Proposition 2.4. *The automorphism l_j and the skew derivation δ_j satisfy the following relation*

$$l_j \delta_j = r^{\langle \mathbf{a}_j, \mathbf{a}_j \rangle} s^{-\langle \mathbf{a}_j, \mathbf{a}_j \rangle} \delta_j l_j.$$

□

Theorem 2.6. *The two-parameter Ringel-Hall algebra $H_{r,s}(A_\infty)$ can be presented as an iterated skew polynomial ring.*

□

Let R be a ring. Recall that a two-sided ideal $P \subset R$ is said to be prime if $P \neq R$ and whenever the product AB of two two-sided ideals

A, B of R is contained in P, then at least one of A and B is contained in P. A two-sided ideal $P \subset R$ is called completely prime if $P \neq R$ and whenever the product ab of two elements of R is contained in P, then at least one of the elements a and b is contained in P. In the case of commutative rings, prime ideals are exactly completely prime ideals. In the case of noncommutative rings, a completely prime ideal is a prime ideal; but a prime ideal may not necessarily be a completely prime ideal. We refer the reader to [2] for more details. Concerning prime ideals of the algebra $H_{r,s}(A_\infty)$, we have the following result.

Corollary 2.1. *Suppose the multiplicative group generated by r, s is torsion-free, then any prime ideal of $H_{r,s}(A_\infty)$ is completely prime.*

Proof: Let $P \subset H_{r,s}(A_\infty)$ be a prime ideal of $H_{r,s}(A_\infty)$. Since $H_{r,s}(A_\infty) = \lim_{n \mapsto \infty} H_{r,s}(A_n)$, we have that $P \cap H_{r,s}(A_n) \subset H_{r,s}(A_n)$ is a prime ideal of $H_{r,s}(\Lambda_n)$ for any $n \geq 1$. Let $a, b \in H_{r,s}(A_\infty)$ such that $ab \in P$. Then we can choose $m \in \mathbb{N}$ such that $a, b, ab \in H_{r,s}(A_m)$. By the result in [33], we know that all prime ideals of $H_{r,s}(\Lambda_m), (m \geq 1)$ are completely prime. Therefore, we have that $a \in H_{r,s}(A_m)$ or $b \in H_{r,s}(A_m)$. Hence, the prime ideal P is a completely prime ideal of $H_{r,s}(A_\infty)$. □

2.3. **An algebra isomorphism from $U^+_{r,s}(\mathfrak{sl}_\infty)$ onto $H_{r,s}(A_\infty)$.** In this subsection, we are going to establish a graded algebra isomorphism from the two-parameter quantized enveloping algebra $U^+_{r,s}(\mathfrak{sl}_\infty)$ onto the two-parameter Ringel-Hall algebra $H_{r,s}(A_\infty)$. Via this isomorphism, all results established in the previous subsection on $H_{r,s}(A_\infty)$ can be transformed to the two-parameter quantized enveloping algebra $U^+_{r,s}(\mathfrak{sl}_\infty)$. Indeed, the isomorphism from $U^+_{r,s}(\mathfrak{sl}_\infty)$ onto $H_{r,s}(A_\infty)$ is the direct limit of the isomorphisms from $U^+_{r,s}(\mathfrak{s}l_{n+1})$ onto $H_{r,s}(A_n)$.

First of all, one can prove the following result, which induces a homomorphism from $U^+_{r,s}(\mathfrak{sl}_\infty)$ into $H_{r,s}(A_\infty)$.

Lemma 2.2. *Let $\alpha_i \in \mathcal{P}$ correspond to the simple module S_i, then we have the following identities in $H_{r,s}(\Lambda_\infty)$.*

$$\alpha_i^2\alpha_{i+1}^2 - (r+s)\alpha_i\alpha_{i+1}\alpha_i + rs\alpha_{i+1}\alpha_i^2 = 0,$$
$$\alpha_i\alpha_{i+1}^2 - (r+s)\alpha_{i+1}\alpha_i\alpha_{i+1} + rs\alpha_i\alpha_{i+1}^2 = 0,$$

for $i = 1, 2, 3, \cdots$.

Proof: Note that we can regard α_i, α_{i+1} as elements of the two-parameter Ringel-Hall algebra $H_{r,s}(A_{i+1})$, which is a subalgebra of $H_{r,s}(A_\infty)$. By the result in [33], we know that these identities hold in the algebra $H_{r,s}(A_{i+1})$. Therefore, we have proved the result as desired. $\square$

Now we have the following result which relates Ringel-Hall $H_{r,s}(A_\infty)$ to the algebra $U^+_{r,s}(\mathfrak{sl}_\infty)$.

Theorem 2.7. *The map*

$$\eta\colon e_i \longrightarrow \alpha_i$$

extends to a $\mathbb{Q}(r,s)-$*algebra isomorphism*

$$\eta : U^+_{r,s}(\mathfrak{sl}_\infty) \longrightarrow H_{r,s}(A_\infty).$$

Proof: (The proof is essentially borrowed from [24] and we include it for completeness. See also [33]). First of all, note that the quantum Serre relations of $U^+_{r,s}(\mathfrak{sl}_\infty)$ are preserved by the map η. Thus the map η does defines an algebra homomorphism from the two-parameter quantized enveloping algebra $U^+_{r,s}(\mathfrak{sl}_\infty)$ into the two-parameter twisted Ringel-Hall algebra $H_{r,s}(A_\infty)$. Now it suffices to show that the map η is indeed a bijection.

We first show that the map η is surjective by verifying that the algebra $H_{r,s}(A_\infty)$ is generated by the elements u_i which correspond to the irreducible representation S_i of the infinite quiver A_∞. Let u_α be any element in $H_{r,s}(A_\infty)$, then we can regard u_α as an element of a certain subalgebra $H_{r,s}(A_n)$. Thus we can restrict our proof to the subalgebra $H_{r,s}(A_n)$. As a result, we have the following:

$$u_\alpha = (\prod_{i=1}^{m} \frac{1}{[\alpha(i)]^!_{\epsilon(\mathbf{a}_i)}}) u_{\mathbf{a}_1}^{\alpha(\mathbf{a}_1)} \cdots u_{\mathbf{a}_m}^{\alpha(\mathbf{a}_m)}.$$

Now we need to prove that u_α is generated by u_i for any α corresponding to an indecomposable representations. We prove this claim by using induction. Note that $\zeta(\alpha,\alpha) = 0$, thus we have the following

$$u_\alpha = u_1^{d_1} \cdots u_n^{d_n} - \sum_{\beta\neq\alpha\ \underline{dim}\beta=\underline{dim}\alpha} s^{\langle\beta,\beta\rangle} u_\beta.$$

However, one sees that the dimension of the module u_β is less than the dimension of the module u_α. Thus by induction on the dimension, we can reduce to the case where $dim(u_\alpha) = 1$. In this case, the only possibility is that $u_\alpha = u_i$ for some i. Thus we have proved the statement that every u_α is generated by u_i, which further implies that the map η is a surjective map. We also note that the map η is a graded map.

Finally, we show that the map η is also injective. Recall that $\mathcal{A} = \mathbb{Q}[r,s]_{(r-1,s-1)}$ denote the localization of the polynomial ring $\mathbb{Q}[r,s]$ at the maximal ideal $(r-1, s-1)$. Then we know that $\mathcal{A} = \mathbb{Q}[r,s]_{(r-1,s-1)}$ is a local ring with the residue field $\mathbb{Q}$ and the fractional field $\mathbb{Q}(r,s)$. Let $U^+_{\mathcal{A}}$ denote the free $\mathcal{A}-$algebra generated by the generators e_i subject to the quantum Serre relations holding in $U^+_{r,s}(\mathfrak{sl}_\infty)$. Also let $U^+_{\mathbb{Q}}(\mathfrak{sl}_\infty)$ denote the universal enveloping algebra of the corresponding nilpotent Lie subalgebra $\mathfrak{n}^+$ of $\mathfrak{s}l_\infty$ defined over the base field $\mathbb{Q}$. Then we have the following

$$U^+_{r,s}(\mathfrak{sl}_\infty) = \mathbb{Q}(r,s) \otimes_{\mathcal{A}} U^+_{\mathcal{A}}, \quad U^+_{\mathbb{Q}}(\mathfrak{sl}_\infty) = \mathbb{Q} \otimes_{\mathcal{A}} U^+_{\mathcal{A}}.$$

For any $\beta \in \mathbb{Z}^{\oplus \mathbb{N}}$, we have the following result via Nakayama's Lemma

$$\begin{aligned} dim_{\mathbb{Q}} U^+_{\mathbb{Q}}(\mathfrak{sl}_\infty)_\beta &= dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathcal{A}} U^+_{\mathcal{A}}))_\beta \\ &\geq dim_{\mathbb{Q}(r,s)}(\mathbb{Q}(r,s) \otimes_{\mathcal{A}} U^+_{\mathcal{A}})_\beta \\ &= dim_{\mathbb{Q}(r,s)} U^+_{r,s}(\mathfrak{sl}_\infty)_\beta \\ &\geq dim_{\mathbb{Q}(r,s)} H_{r,s}(A_\infty)_\beta. \end{aligned}$$

Using **Corollary 2** in the reference [28] and the PBW-theorem, we also have the following result:

$$dim_{\mathbb{Q}} U^+_{\mathbb{Q}}(\mathfrak{sl}_\infty)_\beta = dim_{\mathbb{Q}(r,s)} H_{r,s}(A_\infty)_\beta.$$

Thus we have proved that the map η is injective. Therefore, the map η is an algebra isomorphism from $U^+_{r,s}(\mathfrak{sl}_\infty)$ onto $H_{r,s}(A_\infty)$ as desired. □

□

Based on the previous theorem, the following corollary is in order.

Corollary 2.2. *The algebra $U^+_{r,s}(\mathfrak{sl}_\infty)$ has a $\mathbb{Q}(r,s)-$basis parameterized by the isomorphism classes of all finite dimensional representations of the infinite quiver A_∞. In particular, we have*

$$U^+_{r,s}(\mathfrak{s}l_\infty) = \lim_{n \mapsto \infty} U^+_{r,s}(\mathfrak{s}l_{n+1}).$$

□

Corollary 2.3. *All prime ideals of $U^+_{r,s}(\mathfrak{sl}_\infty)$ are completely prime under the condition that the multiplicative group generated by r,s is torsion-free.*

□

3. The extended two-parameter Ringel-Hall algebras $\overline{H_{r,s}(A_\infty)}$

For the purpose of realizing the Borel subalgebra $U_{r,s}^{\geq 0}(\mathfrak{sl}_\infty)$ of the two-parameter quantum group $U_{r,s}(\mathfrak{sl}_\infty)$, we define the extended Ringel-Hall algebra $\overline{H_{r,s}(A_\infty)}$ by adding the torus part. In particular, we show that there is a Hopf algebra structure on this extended two-parameter Ringel-Hall algebra $\overline{H_{r,s}(A_\infty)}$; as a result we prove that $U_{r,s}^{\geq 0}(\mathfrak{sl}_\infty)$ is isomorphic to the extended two-parameter Ringel-Hall algebra $\overline{H_{r,s}(A_\infty)}$ as a Hopf algebra. Similarly, we can use an extended two-parameter Ringel-Hall algebra to realize the Borel subalgebra $U^{\leq 0}(\mathfrak{s}l_\infty)$. Therefore, we will obtain a PBW-basis of two-parameter quantum group $U_{r,s}(\mathfrak{sl}_\infty)$.

3.1. **Extended Ringel-Hall algebras $\overline{H_{r,s}(A_\infty)}$.** Let us define $\overline{H_{r,s}(A_\infty)}$ to be a free $\mathbb{Q}(r,s)-$module with the following basis

$$\{k_\alpha u_\lambda \mid \alpha \in \mathbb{Z}[I],\, \lambda \in \mathcal{P}\}.$$

Moreover one will define an algebra structure on the module $\overline{H_{r,s}(A_\infty)}$ as follows.

$$u_\alpha u_\beta = \sum_{\lambda\in\mathcal{P}} s^{-\langle\alpha,\beta\rangle} F_{u_\alpha,u_\beta}^{u_\lambda}(rs^{-1})u_\lambda, \quad \text{for any}\, \alpha,\beta\in\mathcal{P},$$

$$k_\alpha u_\beta = r^{\langle\beta,\alpha\rangle} s^{-\langle\alpha,\beta\rangle} u_\beta k_\alpha \quad \text{for any} \quad \alpha\in\mathbb{Z}[I],\beta\in\mathcal{P},$$

$$k_\alpha k_\beta = k_\beta k_\alpha \quad \text{for any} \quad \alpha,\beta\in\mathbb{Z}[I].$$

Indeed, we have the following

Proposition 3.1. *For any elements $x, y, z \in \mathbb{Z}[I]$ and $\alpha,\beta,\gamma\in\mathcal{P}$, we have the following*

$$[(k_x u_\alpha)(k_y u_\beta)](k_x u_\alpha) = (k_x u_\alpha)[(k_y u_\beta)(k_z u_\gamma)].$$

In particular, with the above defined multiplication, $\overline{H_{r,s}(\Lambda_\infty)}$ is an associative $\mathbb{Q}(r,s)-$algebra.

Proof: Once we choose x, y, z, and α,β,γ, we can restrict to the subgroup $\overline{H_{r,s}(A_m)}$ of $\overline{H_{r,s}(A_\infty)}$ for some m. Since $\overline{H_{r,s}(A_m)}$ is an associative algebra with the restricted multiplication, thus we have proved all the statements. □

Furthermore, we have the following result.

Theorem 3.1. *The map η extends to a $\mathbb{Q}(r,s)-$algebra isomorphism from $U_{r,s}^{\geq 0}(\mathfrak{sl}_\infty)$ onto $\overline{H_{r,s}(\Lambda_\infty)}$ via the map $\eta(w_i)=k_i$ and $\eta(e_i)=u_{\alpha_i}$.*

Proof: The proof is straightforward. □

As a result, we have the following description about a basis for the algebra $U^+_{r,s}(\mathfrak{sl}_\infty)$

Corollary 3.1. *The set* $\mathbf{B}^+ = \{w_\alpha \eta^{-1}(u_\lambda) \mid \alpha \in \mathbb{Z}[\mathbb{N}], \lambda \in \mathcal{P}\}$ *is a* $\mathbb{Q}(r,s)-$*basis of* $U^{\geq 0}_{r,s}(\mathfrak{sl}_\infty)$.

□

3.2. **A Hopf algebra structure on** $\overline{H_{r,s}(A_\infty)}$**.** Now we are going to introduce a Hopf algebra structure on the extended two-parameter Ringel-Hall algebra $\overline{H_{r,s}(A_\infty)}$. In particular, we have the following result.

Theorem 3.2. *The algebra* $\overline{H_{r,s}(A_\infty)}$ *is a Hopf algebra with the Hopf algebra structure defined as follows.*

(1) *Multiplication:*

$$u_\alpha u_\beta = \sum_{\lambda\in\mathcal{P}} s^{-\langle\alpha,\beta\rangle} g^\lambda_{\alpha\beta} u_\lambda \quad \text{for any} \quad \alpha,\beta\in\mathcal{B},$$

$$k_\alpha u_\beta = r^{\langle\beta,\alpha\rangle} s^{-\langle\alpha,\beta\rangle} u_\beta k_\alpha \quad \text{for any} \quad \alpha\in\mathbb{Z}[I], \beta\in\mathcal{P},$$

$$k_\alpha k_\beta = k_\beta k_\alpha \quad \text{for any} \quad \alpha,\beta\in\mathbb{Z}[I].$$

(2) *Comultiplication:*

$$\Delta(u_\lambda) = \sum_{\alpha,\beta\in\mathcal{P}} r^{\langle\alpha,\beta\rangle} \frac{a_\alpha a_\beta}{a_\lambda} g^\lambda_{\alpha\beta} u_\alpha k_\beta \otimes u_\beta \quad \text{for any} \quad \lambda\in\mathcal{P},$$

$$\Delta(k_\alpha) = k_\alpha \otimes k_\alpha \quad \text{for any} \quad \alpha\in\mathbb{Z}[I].$$

(3) *Counit:*

$$\epsilon(u_\lambda) = 0 \quad \text{for all} \quad \lambda\neq 0 \quad \text{and} \quad \epsilon(k_\alpha) = 1 \quad \text{for any} \quad \alpha\in\mathcal{P}.$$

(4) *Antipode:*

$$\sigma(u_\lambda) = \delta_{\lambda,0} + \sum_{m\geq 1}(-1)^m \times \sum_{\pi\in\mathcal{P},\lambda_1,\lambda_2,\cdots,\lambda_m\in\mathcal{P}_1} (rs^{-1})^{\sum_{i<j}\langle\lambda_i,\lambda_j\rangle} \frac{a_{\lambda_1}\cdots a_{\lambda_m}}{a_\lambda} g^\lambda_{\lambda_1\cdots\lambda_m} g^\pi_{\lambda_1\cdots\lambda_m} k_{-\lambda} u_\pi$$

for any element $\lambda\in\mathcal{P}$ *and*

$$\sigma(k_\alpha) = k_{-\alpha} \quad \text{for any} \quad \alpha\in\mathbb{Z}[I].$$

In particular, we have the following

$$\overline{H_{r,s}(A_\infty)} = \lim_{n\mapsto\infty} \overline{H_{r,s}(A_n)}$$

as a direct limit of Hopf subalgebras.

□

The proof of the above theorem consists of a couple of lemmas which can be proved as the finite dimensional case. And we refer the reader to [33, 34] for more details. Namely, we have the following lemmas.

Lemma 3.1. *The comultiplication* Δ *is an algebra endomorphism of* $\overline{H_{r,s}(A_\infty)}$.

□

Lemma 3.2. *For any* $\lambda \in \mathcal{P}$, *we have the following*

$$\mu(\sigma \otimes 1)\Delta(u_\lambda) = \delta_{\lambda 0}$$

and

$$\mu(1 \otimes \sigma)\Delta(u_\lambda) = \delta_{\lambda 0}.$$

□

3.3. **A Hopf algebra isomorphism from** $U^{\geq 0}_{r,s}(\mathfrak{sl}_\infty)$ **onto** $\overline{H_{r,s}(A_\infty)}$**.** In this subsection, we will prove that the Borel subalgebras $U^{\geq 0}_{r,s}(\mathfrak{sl}_\infty)$ and $U^{\leq 0}_{r,s}(\mathfrak{sl}_\infty)$ of the two-parameter quantum group $U_{r,s}(\mathfrak{sl}_\infty)$ can be realized as the extended two-parameter Ringel-Hall algebra $\overline{H_{r,s}(A_\infty)}$ and $\overline{H_{s^{-1},r^{-1}}(A_\infty)}$ as Hopf algebras. As a result, we shall derive a PBW-basis for the two-parameter quantum group $U_{r,s}(\mathfrak{sl}_\infty)$.

Theorem 3.3. *We have that*

$$U^{\geq 0}_{r,s}(\mathfrak{sl}_\infty) \cong \overline{H_{r,s}(A_\infty)}$$

and

$$U^{\leq 0}_{r,s}(\mathfrak{sl}_\infty) \cong \overline{H_{s^{-1},r^{-1}}(A_\infty)}$$

as Hopf algebras.

□

Let $\mathbf{B}^-$ denote the $\mathbb{Q}(r,s)-$basis constructed for the algebra $U^{\leq 0}_{r,s}(\mathfrak{sl}_\infty)$ via the Ringel-Hall algebra $\overline{H_{s^{-1},r^{-1}}(A_\infty)}$, then we have the following:

Corollary 3.2. *The set* $\mathbf{B}^+ \times \mathbf{B}^-$ *is a* $\mathbb{Q}(r,s)-$*basis for the two-parameter quantum groups* $U_{r,s}(\mathfrak{sl}_\infty)$.

□

Furthermore, we have the following result, which provides a bridge from the finite dimensional case to the infinite case.

Theorem 3.4. *The two-parameter quantum group* $U_{r,s}(\mathfrak{sl}_\infty)$ *is the direct limit of the direct system* $\{U_{r,s}(\mathfrak{sl}_{n+1}) \mid n \in \mathbb{N}\}$ *of the Hopf subalgebras* $U_{r,s}(\mathfrak{sl}_{n+1})$ *of* $U_{r,s}(\mathfrak{sl}_\infty)$ *. That is*

$$U_{r,s}(\mathfrak{sl}_\infty) = \lim_{n\mapsto\infty} U_{r,s}(\mathfrak{sl}_{n+1}).$$

In particular, we have

$$\begin{aligned} U^{\pm 1}_{r,s}(\mathfrak{sl}_\infty) &= \lim_{n\mapsto\infty} U^{\pm 1}_{r,s}(\mathfrak{sl}_{n+1}), \\ U^{0}_{r,s}(\mathfrak{sl}_\infty) &= \lim_{n\mapsto\infty} U^{0}_{r,s}(\mathfrak{sl}_{n+1}), \\ U^{\geq 0}_{r,s}(\mathfrak{sl}_\infty) &= \lim_{n\mapsto\infty} U^{\geq 0}_{r,s}(\mathfrak{sl}_{n+1}), \\ U^{\leq 0}_{r,s}(\mathfrak{sl}_\infty) &= \lim_{n\mapsto\infty} U^{\leq 0}_{r,s}(\mathfrak{sl}_{n+1}). \end{aligned}$$

Proof: It is obvious that $U_{r,s}(\mathfrak{sl}_{n+1})$ are Hopf subalgebras of $U_{r,s}(\mathfrak{sl}_\infty)$ and $U_{r,s}(\mathfrak{sl}_{m+1})$ for $m \geq n$. In addition, any element of $U_{r,s}(\mathfrak{sl}_\infty)$ is an element of a certain $U_{r,s}(\mathfrak{sl}_{n+1})$. Thus we are done with the proof. □

4. Monomial bases and bar-invariant bases of $U^+_{r,s}(\mathfrak{sl}_\infty)$

In this section, we study various bases of $U^+_{r,s}(\mathfrak{sl}_\infty)$ via the theory of generic extensions. Note that the construction of monomial bases using generic extension theory for the Ringel-Hall algebras of type A, D, E has been done in [9]. The idea of the construction is to use the monoidal structure on the set $\mathcal{M}$ of isoclasses of finite dimensional representations of the corresponding quiver $\mathcal{Q}$ and the Bruhat-Chevalley type partial ordering in $\mathcal{M}$. Note that the arguments used in [9] can be completely transformed to the case of $\mathfrak{sl}_\infty$. Therefore, we will state most of the results for monomial bases without much detail. For the details, we refer the reader to [9, 24].

For the reader's convenience, we will recall the necessary details about the the generic extensions from [9, 24]. Note that there exists a bijective correspondence between the set of positive roots Φ^+ of the root system Φ associated to $\mathfrak{sl}_\infty$ and the set of isoclasses of finite dimensional indecomposable representations of A_∞. For any $\beta \in \Phi^+$, we will denote by $M(\beta) = M_k(\beta)$ the corresponding indecomposable representation of A_∞. By the Krull–Remak-Schmidt theorem, we shall have the following

$$M(\lambda) = M_k(\lambda) \colon = \bigoplus_{\beta\in\Phi^+} \lambda(\beta) M_k(\beta)$$

for some function $\lambda \colon \Phi^+ \longrightarrow \mathbb{N}_0$ with a finite support. Therefore, the isoclasses of finite dimensional representations of A_∞ are indexed by the following set

$$\Lambda = \{\lambda \colon \Phi^+ \longrightarrow \mathbb{N} \text{ with a finite support}\} \cong \mathbb{N}_0^{\oplus \Phi^+}.$$

From now on, we will use the set Λ to index the objects of the category $A_\infty-$mod.

Next, we are going to recall some information about generic extensions of representations of Dynkin quivers. We should mention that all the arguments used in the finite dimensional cases of type A, D, E can be transformed to the $\mathfrak{sl}_\infty$. We refer the interested reader to the references [9, 24] for details.

Let us fix k to algebraically closed. Let us denote by $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$ the quiver A_∞. Fix a $\mathbf{d} = (d_i)_i \in \mathbb{N}_0^{\oplus \Phi^+}$ and we may choose n large enough so that $\mathbf{d}$ can be regarded as an element in $\mathbb{N}_0^n$. For any given $\mathbf{d}$, we can define an affine space as follows

$$R(\mathbf{d}) = R(\mathcal{Q}, \mathbf{d}) \colon = \prod_{\alpha \in \mathcal{Q}_1} Hom_k(k^{d_{t(\alpha)}}, k^{d_{h_\alpha}}) \cong \prod_{\alpha \in \mathcal{Q}_1} k^{d_{t_\alpha} \times d_{h_\alpha}}.$$

Thus, a point $x = (x_\alpha)_\alpha$ of $R(\mathbf{d})$ determines a finite dimensional representation $V(x)$ of $\mathcal{Q} = A_\infty$. The algebraic group $GL(\mathbf{d}) = \prod_{i=1}^n GL_{d_i}(k)$ acts on the space $R(\mathbf{d})$ by the conjugation

$$(g_i)_i \ldots (x_\alpha)_\alpha = (g_{h(\alpha)}) x_\alpha g_{t(\alpha)}^{-1})_\alpha.$$

and the $GL(\mathbf{d})-$orbits $\mathcal{O}_x$ in $R(\mathbf{d})$ correspond bijectively to the isoclasses $[V(x)]$ of finite dimensional representations of $\mathcal{Q}$ with the dimension vector $\mathbf{d}$. The stabilizer $GL(\mathbf{d})_x = \{g \in GL(\mathbf{d}) \mid gx = x\}$ of x is the group of automorphisms of $M \colon = V(x)$ which is zariski-open in $End_{A_n-mod}(M)$ and has a dimension equal to the $dim_{A_n-mod}(M)$. It follows that the orbit $\mathcal{O}_M \colon = \mathcal{O}_x$ of M has a dimension

$$dim\mathcal{O}_M = dimGL(\mathbf{d}) - dimEnd_{A_n-mod}(M).$$

Now we have the following result, whose proof is the same as the one in [24].

Lemma 4.1. *For $x \in R(\mathbf{d_1})$ and any $y \in R(\mathbf{d_2})$, let $\mathcal{E}(\mathcal{O}_x, \mathcal{O}_y)$ be the set of all $z \in R(\mathbf{d})$ where $\mathbf{d} = \mathbf{d_1} + \mathbf{d_2}$ such that $V(z)$ is an extension of some $M \in \mathcal{O}_x$ by some $N \in \mathcal{O}_y$. Then $\mathcal{E}(\mathcal{O}_x, \mathcal{O}_y)$ is irreducible.*

□

Given any two finite-dimensional representations of M, N of the infinite linear quiver A_∞, let us consider the extensions

$$0 \longrightarrow N \longrightarrow L \longrightarrow M \longrightarrow 0$$

of M by N. By the lemma, there is a unique (up to isomorphism) such extension G with $dim\mathcal{O}_G$ being maximal. We call G the generic extension of M by N, and denoted by $M * N$. For any two representations M, N, we say M degenerates to N, or that N is a degeneration of M, and write $[N] \leq [M]$ (or simply $N \leq M$) if $\mathcal{O}_N \subseteq \overline{\mathcal{O}_M}$ which is the closure of $\mathcal{O}_M$. Note that $N < M$ if and only if $\mathcal{O}_N \subset \overline{\mathcal{O}_M} \backslash \mathcal{O}_M$.

Similar to the result in [9, 24], one knows that the relation $\leq$ is independent of the base field k and it provides a partial order on the set Λ via setting $\lambda \leq \mu$ if and only if $M_k(\lambda) \leq M_k(\mu)$ for any given algebraically closed field k.

Using the same arguments as in [9, 24], we shall have the following result.

Theorem 4.1. *(1). If* $0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$ *is exact and non-split, then* $M \oplus N < E$.
(2). Let M, N, X *be finite dimensional representations of the quiver* A_∞. *Then* $X \leq M * N$ *if and only if there exit* $M' \leq M, N' \leq N$ *such that* X *is an extension of* M' *by* N'. *In particular, we have* $M' \leq M, N' \leq N \Longrightarrow M' * N' \leq M * N$.
(3). Let $\mathcal{M}$ *be the set of isoclasses of finite dimensional representations of* A_∞ *and define a multiplication* $*$ *on* $\mathcal{M}$ *by* $[M] * [N] = [M * N]$ *for any* $[M], [N] \in \mathcal{M}$. *Then* $\mathcal{M}$ *is a monoid with identity* $1 = [0]$ *and the multiplication* $*$ *preserves the induced partial ordering on* $\mathcal{M}$.
(4). $\mathcal{M}$ *is generated by irreducible representations* $[S_i], i \in I$ *subject to the following relations*

1. $[E_i] * [E_j] = [E_j][E_i]$ *if* i, j *are not connected by an arrow,*
2. $[E_i] * [E_j] * [E_i] = [E_i] * [E_i] * [E_j]$ *and* $[E_j] * [E_i] * [E_j] = [E_i] * [E_j] * [E_j]$ *if there exists an arrow from* i *to* j.

□

In addition, let us denote by U_q^+ the $\mathbb{Q}[q]-$algebra generated by $E_i, i \in I$ subject to the relations

1. $E_iE_j = E_jE_i$ if i, j are not connected by an arrow,
2. $E_i^2E_j - (q+1)E_iE_jE_i + qE_jE_i^2 = 0$ and $E_iE_j^2 - (q+1)E_jE_iE_j + qE_j^2E_i = 0$ if there exists an arrow from i to j.

Let $H_q(A_\infty)$ denote by the generic Ringel-Hall algebra defined over $\mathbb{Q}[q]$ associated to the infinite quiver A_∞.

Adopting the same argument in [24], we shall further have the following result.

Theorem 4.2. *The following algebras are isomorphic.*

1. *The monoid ring* $\mathcal{M}$.

(2) *The* $\mathbb{Q}-$*algebra with generators* $i \in I$ *and relations*
   (a) $ij = ji$ *if* i *and* j *are connected by arrow,*
   (b) $iji = iij$ *and* $jij = ijj$ *if there is an arrow from* i *to* j*.*
(3) *The specialization* U_0^+ *of* U_q^+ *to* $q = 0$*.*
(4) *The specialization* $H_0(A_\infty)$ *of* $H_q(A_\infty)$ *to* $q = 0$*.*

□

Let us denote by Ω the set of all words formed by letters in I. It is easy to see that for any given word $w = w_1 \cdots w_m \in \Omega$, we can set the following finite dimensional representations of A_∞

$$M(w) = S_{w_1} * S_{w_2} * \cdots * S_{w_m}.$$

Note that there is a unique $M(\mathfrak{p}(w)) \in A_\infty - mod$ such that $M(w) \cong M(\mathfrak{p}(w))$, which enables us to define a function as follows

$$\mathfrak{p} \colon \Omega \longrightarrow A_\infty - mod, w \mapsto M(\mathfrak{p}(w)).$$

Furthermore, we shall have the following result on this function.

Theorem 4.3. *The map* $\mathfrak{p}$ *induces a surjection*

$$\mathfrak{p} \colon \Omega \longrightarrow A_\infty - mod.$$

Proof: Once again, we can restrict the function to a certain subcategory A_m-mod, where the property holds. □

Therefore, $\mathfrak{p}$ induces a partition of the set $\Omega = \cup_{\lambda \in \Lambda} \Omega_\lambda$ with $\Omega_\lambda = \mathfrak{p}^{-}(\lambda)$. We will call each Ω_λ a fiber of the map $\mathfrak{p}$.

Now we are going to recall some information on the partial ordering $\leq$. Let $w = i_1 \cdots i_m$ be a word in Ω. Then w can be uniquely expressed in the tight form $w = j_1^{e_1} \cdots j_t^{e_t}$ where $e_r \geq 1, 1 \leq r \leq t$, and $j_r \neq j_{r+1}$ for $1 \leq r \leq t-1$. A filtration

$$0 = M_t \subset M_{t-1} \subset \cdots M_1 \subset M_0 = M$$

of a module M is called a reduced filtration of type w if $M_{r-1}/M_r \cong e_r S_r$, for all $1 \leq r \leq t$. Note that any reduced filtration of M of type w can be refined to a composition of M of type w. Conversely, given a composition series of M, there is a unique reduced filtration of M. Let us denote by $\varphi_w^\lambda(x)$ the Hall polynomial $\varphi_{\mu_1 \cdots \mu_t}^\lambda(x)$ where $M(\mu_r) = e_r S_r$. Let us denote by $\gamma_w^\lambda(q_k)$ the number of the reduced filtrations of $M_k(\lambda)$ over the base field k when k is a finite field. A word w is called distinguished if $\gamma_w^{\mathfrak{p}(w)} = 1$. Note that w is distinguished if and only if, for some algebraically closed field k, $M_k(\mathfrak{p}(w))$ has a unique reduced filtration of type w. Similar to [9], we have the following results.

Lemma 4.2. *(See also* **Lemma 4.1** *in* [9]*) Let $\omega \in \Omega$ and $\mu \geq \lambda$ in Λ. Then $\varphi_\omega^\mu \neq 0$ implies $\varphi_\omega^\lambda \neq 0$.*

□

Theorem 4.4. *(See also* **Theorem 4.2** *in* [9]*) Let $\lambda, \mu \in \Lambda$. Then $\lambda \leq \mu$ if and only if there exists a word $\omega \in \mathfrak{p}^{-1}(\mu)$ such that $\varphi_\mu^\lambda \neq 0$.*

□

Lemma 4.3. *(See also* **Lemma 5.2** *in* [9]*) Every fiber of $\mathfrak{p}$ contains a distinguished word.*

□

Let us define $[[e_a]]^! = [[1]] \cdots [[e_a]]$ with $[[m]] = \frac{1-(rs^{-1})^m}{1-rs^{-1}}$. Then we shall have the following result.

Lemma 4.4. *(See also* **Lemma 6.1** *in* [9]*) Let $w \in \Omega$ be a word with the tight form $j_1^{e_1} \cdots j_t^{e_t}$. Then, for each $\lambda \in \Lambda$,*

$$\varphi_w^\lambda(rs^{-1}) = \gamma_w^\lambda(rs^{-1}) \prod_{r=1}^{t} [[e_r]]^!.$$

In particular, $\varphi_w^{\mathfrak{p}(w)} = \prod_{r=1}^{t} [[e_r]]^!$ if w is distinguished.

□

For any given word $w = i_1 \cdots c_m \in \Omega$, we can associate a monomial

$$u_w = u_{i_1} \cdots u_{i_m} \in H_{r,s}(A_\infty).$$

Proposition 4.1. *For any $w \in \Omega$ with the tight form $j_1^{r_1} \cdots j_t^{e_t}$, we have*

$$u_w = \sum_{\lambda \leq \mathfrak{p}(w)} \varphi_w^\lambda(rs^{-1}) u_\lambda = \prod_{r=1}^{t} [[e_r]]^! \sum_{\lambda \leq \mathfrak{p}(w)} \gamma_w^\lambda(rs^{-1}) u_\lambda.$$

Moreover, the coefficients appearing in the sum are all nonzero.

□

As a result, we shall have the following theorem.

Theorem 4.5. *For each given $\lambda \in \Lambda$, let us choose an arbitrary word $w_\lambda \in \mathfrak{p}^{-1}(w)$. Then the set $\{u_{w_\lambda} \mid \lambda \in \Lambda\}$ is a $\mathbb{Q}(r,s)-$basis of $H_{r,s}(A_\infty)$. Moreover, if all the words are chosen to be distinguished, then this set is a $\mathbb{Z}[r,s]_{(r-1,s-1)}-$basis of $H_{r,s}(A_\infty)_{\mathbb{Z}[r,s]_{(r-1,s-1)}}$.*

□

4.1. **A bar-invariant basis of $U^+_{r,s}(\mathfrak{s}l_\infty)$.** It is easy to see that the algebra $U^+_{r,s}(\mathfrak{sl}_\infty)$ admits a $\mathbb{Q}-$linear involution defined as follows

$$\overline{r}=s, \overline{s}=r, \overline{e_i}=e_i \text{ for all } i\in I.$$

And we will refer this involution as the bar-involution. In this subsection, we will construct a bar-invariant basis for $U^+_{r,s}(\mathfrak{sl}_\infty)$.

Denote by $[M,N]=dim_k Hom(M,N)$ and $[M,N]^1=dim_k Ext^1(M,N)$. Let us set $c^X_{M,N}=s^{[X,X]-[M,N]+[M,N]^1-[M,M]-[N,N]}F^X_{M,N}(rs^{-1})$. It is obvious that the same proof in [24] shall yield the following result.

Proposition 4.2. *Let us write* $\overline{u_\alpha}=\sum_\beta \omega^\alpha_\beta u_\beta$, *then we have*

(1) $\omega^\alpha_\beta=0$ *unless* $\beta\le\alpha$, *and* $\omega^\alpha_\alpha=1$,

(2) *if* $u_\alpha=M\oplus N$ *for finite dimensional representations* M,N *with* $[N,M]=0=[M,N]^1$, *then*

$$\omega^\alpha_\beta=\sum_{M'\le M, N'\le N}\omega^M_{M'}\omega^N_{N'}c^\alpha_{M'N'},$$

(3) *if* u_α *is an exponent of a finite dimensional indecomposable representation, then*

$$\omega^\alpha_\beta=s^{[u_\beta,u_\beta]^1}-\sum_{\beta\le\gamma<\alpha}r^{[u_\gamma,u_\gamma]^1}\omega^\gamma_\beta,$$

(4) $\omega^\alpha_\beta\in s^{[u_\beta,u_\beta]-[u_\alpha,u_\alpha]}\mathbb{Z}[rs^{-1}]$.

□

Furthermore, using the arguments in [21, 24], we shall have the following result on a bar-invariant basis of the algebra $U^+_{r,s}(\mathfrak{s}l_\infty)$.

Theorem 4.6. *For each isoclass* α, *there exists a unique element*

$$\mathcal{C}_\alpha\in u_\alpha+s^{-1}\mathbb{Z}[rs,r^{-1}s^{-1},s][\mathbf{B}\backslash\{u_\alpha\})$$

such that $\overline{\mathcal{C}_\alpha}=\mathcal{C}_\alpha$. *Write* $\mathcal{C}_\alpha=\sum_\beta\zeta^\alpha_\beta u_\beta$, *we have*

(1) $\zeta^\alpha_\beta=0$ *unless* $\beta\le\alpha$, *and* $\zeta^\alpha_\alpha=1$,

(2) $\zeta^\alpha_\beta\in s^{[u_\beta,u_\beta]-[u_\alpha,u_\alpha]}\mathbb{Z}[rs^{-1}]$,

(3) *Denote by* $\hat{\zeta}^\alpha_\beta(v)\in\mathbb{Z}[v,v^{-1}]$ *the specialization of* ζ^α_β *to* $\alpha=v=s^{-1}$, *we have*

$$\zeta^\alpha_\beta=(\sqrt{rs})^{[u_\beta,u_\beta]-[u_\alpha,u_\alpha]}\hat{\zeta}^X_Y(\sqrt{rs^{-1}}).$$

□

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