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# Generalized Necessary and Sufficient Conditions for Annihilation of HIV-1 Virions During HAART

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**Abstract:** In this paper, the patho-physiological dynamics of Human Immuno-deficiency Virus type 1 (HIV-1) induced AIDS during Highly Active Anti Retroviral Therapy (HAART) is modeled by a system of non-linear deterministic differential equations. The physiologically relevant and clinically plausible equations depict the dynamics of uninfected CD4<sup>+</sup> T cells ( $x_1$ ), HIV-1 infected CD4<sup>+</sup> T cells ( $x_2$ ), HIV-1 virions in the blood plasma ( $x_3$ ), HIV-1 specific CD8<sup>+</sup> T cells ( $x_4$ ), and the concentration of HAART drug molecules ( $x_5$ ). Criteria for the existence of therapeutic outcomes are presented. In particular, the necessary and sufficient conditions for the annihilation of HIV-1 virions, and HIV-1 infected helper T cells are clearly exhibited in terms of biological measurable model physiological parameters. Investigative computer simulations are presented elucidating the patho-physiodynamics of HIV-1 induced AIDS and various hypothetical patient parametric configurations. The mathematical analysis of the model equations and the computer simulations are performed with regard to HAART protocols with constant continuous intravenous and transdermal drug infusions.

**Keywords:** HIV-1 patho-physiodynamics, mathematical modeling, HAART therapy, AIDS cure criteria, Michaelis-Menten kinetics

**AMS Subject Classification:** 93A30; 93D05; 93D20; 34A34; 92C42; 92C35

## 1. Introduction

Highly Active Anti-retroviral Therapy (HAART) is currently the most therapeutically efficacious treatment protocol for treating the Acquired Immunodeficiency Syndrome (AIDS). HIV-1 virions induce AIDS by orchestrating an irreversible destruction of the CD4<sup>+</sup> T cells which then paralyze the immune system of the HIV-1 positive person. The major objectives of HAART therapy are the prolongation and improvement of the long-term life quality of patients; optimization of therapy such as to suppress the HIV-1 viral load to below 50 copies of HIV-1 RNA; reconstitution of the patients' immune system such that the CD4<sup>+</sup> T cells proliferate to carrying capacity; and minimization of drug toxicity. HAART treatment protocol consists of nucleoside reverse transcriptase inhibitors, non-nucleoside reverse transcriptase inhibitors, protease inhibitors, anti-fungals /anti-bacterials and in future, integrase inhibitors. The reverse transcriptase inhibitors prevent reverse transcription of HIV-1 specific DNA. The protease inhibitors are antagonistic to maturation and formation of new HIV-1 virions. The possible role of integrase inhibitors is to prevent the integration of HIV-1 viral DNA into the patients' DNA[5,12].

In order to achieve the therapeutic goals of HAART, it is plausible to involve the techniques of mathematical modeling. Before the advent of HAART, the primary focus of the mathematical modelers is to quantitatively analyze the observed patho-physio dynamics of HIV-1 infection in the AIDS patients. These earlier research papers involve the pioneering work of Perelson et al. [13], Nowark et al [11], and other contributors [16,17].

A recent paper by Nani and Jin [10] provided some physiological criteria under which HIV-1 virions in an AIDS patient can be annihilated during HAART.

Some of the earlier mathematical modeling publications focus on single-drug AIDS therapy using Zidovudine (INN) or azidothymidine (AZT) [9]. Then the advent of Active Retro-viral Therapy (ART) and the associated clinical limitation led to the development of HAART treatment protocols. In spite of the initial success of HAART, there are clinically measurable and observable shortcomings in the treatment of AIDS [4,]. In particular, HAART is not successful in about 40% of AIDS patients because of drug-induced toxicity and complications of treatment. HAART protocols have been clinically observed to have limited therapeutic efficacy due to biochemical/clinical drug resistance, short drug half-life, low bio-availability and blood plasma toxicities.

Mathematical modeling provides a quantitative and rational approach to solve the therapeutic efficacy problems associated with HAART. In particular, the models focused on finding optimal therapeutic schedules, the roles of latent viral reservoirs as well as minimizing of toxic side effect[1,2,6,7,8,9,14,16,17].

43 Optimal therapies that will minimize side effects have been investigated by many authors in [8, 9, 10, 11, 13, 14, 15, 16].  
 44 Zaric et al. in 1998 presented a model which was focused on the simulation of protease inhibitors and role of drug resistant  
 45 HIV-1 virions [18]. Stengel in [14] presented a mathematical model of HIV-1 infection and HAART which demonstrated the  
 46 efficacy of a mathematically optimal therapy. Caetano and Yoneyama in [2] constructed a HAART model which incorporated  
 47 the roles of latently infected  $CD4^+$  T cells, and discussed how the reverse transcriptase and protease inhibitors affected HIV-1  
 48 dynamics during HAART, using the LQR, Scheme. In a future paper, we will use Pontryagin's Minimal Principle to construct  
 49 admissible optimal therapies such as to minimize the toxicity of the drug but maximize the therapeutic efficacy of HAART.

50 In this paper, an elaborate mathematical model will be constructed which will incorporate physiologically plausible effects  
 51 such as Michaelis-Menten kinetics, role of HIV-1 latent viral reservoirs, continuous transdermal drug delivery, and the implicit  
 52 lymphocyte proliferation induction by the  $CD4^+$  T cells. The activation and proliferation is accomplished by a paracrine and  
 53 autocrine processes which are mediated by the cytokine interleukin-2, secreted by the  $CD4^+$  T cells. Several authors  
 54 investigated the consequences of structured long-term and short-term treatment interruptions during HAART [1, 2, 4, 8]. The  
 55 current model will discuss these consequences by means of simulations.

56 The current paper will be divided into seven sections. The first section gives the introduction into HAART therapy and  
 57 provides the basis for current research. This is followed by presentation and discussion of the model parameters in Section 2. In  
 58 Section 3 the mathematical model of HAART therapy will be constructed. Also the necessary and sufficient criteria for  
 59 annihilation of HIV-1 virions during HAART will be presented in sections 4, 5, and 6. In Section 7, clinically plausible  
 60 computer simulations will be exhibited. Section 5 will be the summary and discussion of the basic results of the paper.

61

## 62 2. Parameters

63 In this section, the physiological variable and parameters of the HAART model equations will be defined and explained. It  
 64 must be emphasized that some of these parameters are biologically measurable or can be estimated using clinical techniques. In  
 65 clinical experience, these parameters are different from patient to patient depending on their patho-physiological conditions.

66 A list of model parameters, constants, and variables is shown as follows.

67  $x_1$ : the number density of non-HIV-1-infected  $CD4^+$  helper T-lymphocytes per unit volume

68  $x_2$ : the number density of HIV-1 infected  $CD4^+$  helper T-lymphocytes per unit volume

69  $x_3$ : the number density of HIV-1 virions in the blood plasma per unit volume

70  $x_4$ : the number density of HIV-1 specific  $CD8^+$  cytotoxic T-lymphocytes per unit volume

71  $x_5$ : the concentration of drug molecules of the HAART treatment protocol

72  $S_1$ : rate of supply of un-infected  $CD4^+$   $T_4$ -lymphocytes

73  $S_2$ : rate of supply of latently infected  $CD4^+$   $T_4$ -lymphocytes

74  $S_3$ : rate of supply of HIV-1 virions from macrophage, monocytes, microglial cells and other lymphoid tissue different from  
 75  $T_4$ -lymphocytes

76  $S_4$ : rate of supply of  $CD8^+$   $T_8$  lymphocytes from the thymus

77  $D$ : rate of HAART drug infusion by transdermal delivery

78  $a_i, b_i$ : constant associated with activation of lymphocytes by cytokine interleukin-2 (IL-2) ( $i = 1, 2, 3, 4$ )

79  $c$ : rate of HAART drug degradation and excretion

80  $\alpha_i$ : constant associated with HIV-1 infection of  $CD4^+$   $T_4$  helper cells ( $i = 1, 2, 3$ )

81  $\beta_1$ : the number of HIV-1 virions produced per day by replication and budding in  $CD4^+$   $T_4$  helper cells

82  $\beta_2$ : rate constant associated with replication and "budding" of HIV-1 in syncytia  $CD4^+$   $T_4$  helper cells per day per microliter ( $\mu l$ )  
 83 and released into the blood plasma

84  $\beta_3$ : the number of HIV-1 virions produced per day by replication and "budding" in non-syncytia  $CD4^+$   $T_4$  helper cells and  
 85 released into the blood plasma

86  $\eta_i$ : constant depicting the rate of which HIV-1 virions incapacitate the  $CD8^+$   $T_8$  cytotoxic cells ( $i = 1, 2$ )

87  $(\sigma_0, \lambda_0)$ : Michaelis-Menten nonlinear metabolic rate constants associated with HAART drug elimination

88  $(\sigma_i, \lambda_i)$ : Michaelis-Menten nonlinear metabolic rate constants associated with HAART drug pharmacokinetics ( $i = 2, 3$ )

- 89  $\xi_i$ : cytotoxic coefficient where  $0 \leq \xi_i \leq 1$  ( $i = 2, 3$ )  
90  $q_i$ : constant depicting competition between infected and un-infected  $CD4^+$  T<sub>4</sub> helper cells ( $i = 1, 2$ )  
91  $k_i$ : constant depicting degradation, loss of clonogenicity or “death” ( $i = 1, 2, 3, 4$ )  
92  $k_5$ : rate constant depicting linear drug elimination pharmacokinetics  
93  $e_{10}$ : constant depicting death or degradation or removal by apoptosis (programmed cell death) ( $i = 1, 2, 3, 4$ )  
94  $K_i$ : constant associated with the killing rate of infected  $CD4^+$  T<sub>4</sub> cells by  $CD8^+$  T<sub>8</sub> cytotoxic lymphocytes ( $i = 1, 2$ )  
95 All the parameters are positive  
96

### 97 3. Model Equations

#### 98 3.1. Description of the Model Equations

99 The HIV-1 patho-physiological dynamics during HAART therapy can be modeled using the following system of  
100 non-linear ordinary differential equations:  
101

$$\begin{cases}
\dot{x}_1 = S_1 + a_1 x_1^2 e^{-b_1 x_1} - \alpha_1 x_1 x_3 - q_1 x_1 x_2 - k_1 x_1 - e_{10} \\
\dot{x}_2 = S_2 + a_2 x_1 x_2 e^{-b_2 x_1} + \alpha_2 x_1 x_3 - q_2 x_1 x_2 - k_2 x_2 - \beta_1 x_3 \\
\quad - K_1 x_2 x_4 - e_{20} - \frac{\xi_2 \sigma_2 x_2 x_5}{\lambda_2 + x_5} \\
\dot{x}_3 = S_3 + \beta_2 x_2 x_3 + \beta_3 x_3 - \alpha_3 x_1 x_3 - \eta_1 x_3 x_4 - k_3 x_3 - e_{30} \\
\quad - \frac{\xi_3 \sigma_3 x_3 x_5}{\lambda_3 + x_5} \\
\dot{x}_4 = S_4 + a_4 x_1 x_4 e^{-b_4 x_1} - K_2 x_2 x_4 - \eta_2 x_3 x_4 - k_4 x_4 - e_{40} \\
\dot{x}_5 = Df(t) - \frac{\sigma_0 x_5}{\lambda_0 + x_5} - \frac{\sigma_2 x_2 x_5}{\lambda_2 + x_5} - \frac{\sigma_3 x_3 x_5}{\lambda_3 + x_5} - k_5 x_5 \\
f(t) = \begin{cases} 1 & \text{for constant continuous input} \\ |\lceil \sin nt \rceil| & \text{for periodic input} \end{cases} \\
x_i(t_0) = x_{i0} \quad \text{for } i = \{1, 2, 3, 4, 5\}
\end{cases} \quad (3.1)$$

113 The model includes the following clinical improvements:

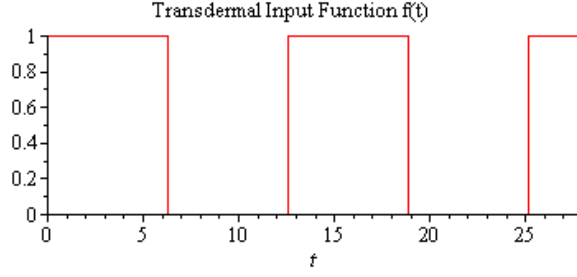
114 (i) The drug delivery uses transdermal, stealth-liposome encapsulated drug delivery, instead of the matrix tablet form because  
115 of improved therapeutic efficacy and reduced gastro-intestinal toxicity [6]. It is also assumed that elastic liposomes are  
116 formulated and selectively targeted such as to reduce toxicity to non-HIV-1-infected  $CD4^+$  T cells ( $x_1$ ) and  $CD8^+$  cytotoxic  
117 T cells ( $x_4$ ).

118 (ii) The HAART drug is such that each renal excretion and body clearance rate follows Michaelis-Menten kinetics.

119 (iii)  $g(x_1, x_j) = a_j x_1 x_j e^{-b_j x_1}$  for  $j = (1, 2, 4)$

120 This function depicts the process of lymphocyte activation which is mediated by  $x_1$  ( $CD4^+$ ) T helper cells. These cells  
121 secrete a cytokine called interleukin-2.

122 (iv) The periodic input function  $f(t) = |\lceil \sin(5t) \rceil|$  can be depicted by the following plot:  
123



124

125

### 126 3.2. Boundedness and Invariance of Non-negativity of Solutions

127 In this subsection theoretical conditions will be constructed under which solutions to the HAART mathematical model  
 128 equations are well-posed, ultimately bounded, and exhibit invariance of non-negativity for all  $t \in [t_0, T] \subset \mathfrak{R}_+ = [0, \infty)$ . In this  
 129 case,  $t_0$ , and  $T$  are defined respectively as times at which HAART therapy begins and terminates.

130 **Theorem 3.1** Consider

131 (i)  $\Omega = \left\{ (x_1, x_2, x_3, x_4, x_5) \in \mathfrak{R}_+^5 \mid 0 \leq x_i \leq \Phi_i, i = 1, 2, 3, 4, 5 \right\}$   
 132 where  $\Phi_i = \sup_{t \in [t_0, T]} x_i$

133 Let

134  $\Phi_i = \max \left\{ (x_{i0}, \frac{S_i + C_i - e_{i0}}{k_i}), i = 1, 2, 3, 4 \right\}$

135 (ii)

136  $\Phi_5 = \max \left\{ x_{50}, \frac{D}{\delta} \right\}$

137 Then there exists a  $T_0 > 0$  such that for  $T_0 < t < \infty$ , all solutions to the HAART model equations (3.1) with initial values  $x_{i0} \in \mathfrak{R}_+^5$   
 138  $= \{x_i \in \mathfrak{R} \mid x_i \geq 0, i = (1, 2, 3, 4, 5)\}$  are ultimately bounded, dissipative, and will eventually enter the non-negatively invariant region

139  $\Omega$ . In particular, the solutions are trapped in the region  $\Omega$  for all  $t > T_0 \subset \mathfrak{R}_+$

140 **Proof.** Using the result from Nani and Jin in [10], let

141  $C_j = \sup_{t \in [t_0, T]} [a_j x_1 x_j e^{-b_j x_1}]$  for  $j = \{1, 2, 4\}$  (3.2)

142  $C_3 = \sup_{t \in [t_0, T]} [\beta_2 x_2 x_3 + \beta_3 x_3]$

143 Define

144  $\delta_i = \sup_{t \in [t_0, T]} \left[ \frac{1}{\lambda_i + x_5} \right]$  for  $i = \{0, 2, 3\}$  (3.3)

145 And set

146  $\delta = \delta_0 + \delta_2 + \delta_3$  (3.4)

147 The Kamke comparison theorem, cf. [10] can be used to establish the following inequalities.

148 
$$\begin{cases} x_i \leq \frac{S_i + C_i - e_{i0}}{k_i} + \gamma_i e^{-k_i t} & \text{for } i = \{1, 2, 3, 4\} \\ x_5 \leq \frac{D}{\delta} + \gamma_5 e^{-\delta t} & \text{where } \delta = \delta_0 + \delta_2 + \delta_3 \\ \text{and } \gamma_i \in \mathfrak{R}_+ = (0, \infty) & \text{and } i = \{1, 2, 3, 4, 5\} \end{cases}$$
 (3.5)

152 In particular, the following results can be obtained.

$$\limsup x_i(t) \leq \frac{S_i + C_i - e_{i0}}{k_i}, \quad i = \{1, 2, 3, 4\}$$

$$\limsup x_5(t) \leq \frac{D}{\delta}$$

153

154

$$(3.6)$$

155

156 Thus,  $\Omega$  is non-negatively invariant and the system is ultimately bounded, dissipative, with the bounds defined by the following  
157 equations.

158

$$\sup_{t \in [t_0, T]} x_i(t) = \max_{t \in [t_L, t_P]} \left\{ x_{i0}, \frac{S_i + C_i - e_{i0}}{k_i} \right\}, \quad i = \{1, 2, 3, 4\}$$

159

$$\sup_{t \in [t_0, T]} x_5(t) = \max_{t \in [t_0, T]} \left\{ x_{50}, \frac{159}{\delta} \right\} \tag{3.7}$$

160

161

This completes the proof.  $\square$

162

## 4. The Rest Points and Computation of the Jacobian Matrices

163

### 4.1 The list of rest points or physiological outcomes of HAART

164

In this section, the possible patho-physiological outcomes from constant continuous transdermal HAART therapy are  
165 listed and analyzed.

166

The physiological outcomes or the steady states during constant continuous transdermal HAART therapy occur when

167

$$\dot{x}_i = 0 \text{ for } i=1,2,3,4,5 \text{ and } f(t) \equiv 1$$

168

In particular, the clinically relevant and physiological plausible therapeutic outcomes include the following:

169

$$E_1 = [0, 0, 0, 0, x_5]$$

170

$$E_2 = [0, 0, 0, x_4, x_5]$$

171

$$E_3 = [x_1, 0, 0, x_4, x_5]$$

172

$$E_4 = [x_1, 0, 0, 0, x_5]$$

173

$$E_5 = [0, x_2, x_3, 0, x_5]$$

174

$$E_6 = [0, 0, x_3, 0, x_5]$$

175

$$E_7 = [0, x_2, 0, 0, x_5]$$

176

$$E_8 = [x_1, x_2, 0, 0, x_5]$$

177

$$E_9 = [x_1, x_2, 0, x_4, x_5]$$

178

$$E_{10} = [x_1, x_2, x_3, x_4, x_5]$$

179

There are some other steady states which are not listed because they are less clinically interesting.

180

The clinically desirable steady states for a HIV-1 AIDS patient are  $E_3$  and  $E_4$ . The steady state  $E_{10}$  depicts a person who is  
181 living with AIDS. In this case, the model exhibits persistence and the viral titer is not sufficient to annihilate the immune system.  
182 The steady states  $E_1$  and  $E_2$  represent scenarios of therapeutic failure because the  $CD4^+$  T cells ( $x_1$ ) are obliterated by the  
183 cytotoxicity of the HAART protocol. On the other hand,  $E_5$  represents the scenario in which HIV-1 virions ( $x_3$ ), HIV-1 infected  
184  $CD4^+$  T cells ( $x_2$ ), and the HAART drug ( $x_5$ ) eliminate the uninfected  $CD4^+$  T cells ( $x_1$ ), and HIV-1 specific  $CD8^+$  T cells ( $x_4$ ).  
185 This is also an example of therapeutic failure for HAART protocol. In  $E_6$  and  $E_7$ , the uninfected  $CD4^+$  T cells are obliterated by  
186 the HAART protocol and consequently are not clinically desirable. The steady states  $E_8$  and  $E_9$  represent curious scenarios  
187 because the HIV-1 virions ( $x_3$ ) are eliminated from blood plasma but unfortunately the HIV-1 infected  $CD4^+$  cells ( $x_2$ ) remain  
188 and will constitute a reservoir from which the HIV-1 virions will burst and repopulate the blood plasma and re-infect other  
189 lymphoid organs.

190

The clinically desirable steady states  $E_3$  and  $E_4$  as well as undesirable states  $E_5$  will be discussed in this section. The  
191 mathematical techniques used include the Hartman-Grobman theorem, non-linear dynamic systems theory, and the principles  
192 of linearized stability.

193

194 **4.2 Computation of the Jacobian Matrices**

195 Using the Hartman-Grobman theorem, it is possible to investigate the physiological stability of HIV-1 AIDS disease  
 196 dynamics associated with the model equations, in the neighborhood of the physiological outcomes (steady states).

197 The Jacobian matrix of linearization near any physiological outcome is denoted symbolically by

$$198 \quad J[E_k] := \{a_{ij}\}_{5 \times 5} \in M_{5 \times 5}(\mathfrak{R}) \quad k = 3, 4, 5 \quad (4.1)$$

199 In particular, the  $a_{ij}$  entries are defined as follows:

$$\begin{aligned}
 200 \quad a_{11} &:= a_1 x_1 (2 - b_1 x_1) e^{-b_1 x_1} - \alpha_1 x_3 - q_1 x_2 - k_1 \\
 201 \quad a_{12} &:= -q_1 x_1 \\
 202 \quad a_{13} &:= -\alpha_1 x_1 \\
 203 \quad a_{14} &:= 0 \\
 204 \quad a_{15} &:= 0 \\
 205 \quad a_{21} &:= a_2 x_2 (1 - b_2 x_1) e^{-b_2 x_1} - q_2 x_2 \\
 206 \quad a_{22} &:= a_2 x_1 e^{-b_2 x_1} - q_2 x_1 - k_2 - K_1 x_4 - \frac{\xi_2 \sigma_2 x_5}{\lambda_2 + x_5} \\
 207 \quad a_{23} &:= \alpha_2 x_1 - \beta_1 \\
 208 \quad a_{24} &:= -K_1 x_2 \\
 209 \quad a_{25} &:= -\frac{\xi_2 \lambda_2 \sigma_2 x_2}{(\lambda_2 + x_5)^2} \\
 210 \quad a_{31} &:= -\alpha_3 x_3 \\
 211 \quad a_{32} &:= \beta_2 x_3 \\
 212 \quad a_{33} &:= \beta_2 x_2 + \beta_3 - \alpha_3 x_1 - \eta_1 x_4 - k_3 - \frac{\xi_3 \sigma_3 x_5}{\lambda_3 + x_5} \\
 213 \quad a_{34} &:= -\eta_1 x_3 \\
 214 \quad a_{35} &:= -\frac{\xi_3 \sigma_3 \lambda_3}{(\lambda_3 + x_5)^2} \\
 215 \quad a_{41} &:= a_4 x_4 (1 - b_4 x_1) e^{-b_4 x_1} \\
 216 \quad a_{42} &:= -K_2 x_4 \\
 217 \quad a_{43} &:= -\eta_2 x_4 \\
 218 \quad a_{44} &:= a_4 x_1 e^{-b_4 x_1} - K_2 x_2 - \eta_2 x_3 - k_4 \\
 219 \quad a_{45} &:= 0 \\
 220 \quad a_{51} &:= 0 \\
 221 \quad a_{52} &:= -\frac{\sigma_2 x_5}{\lambda_2 + x_5} \\
 222 \quad a_{53} &:= -\frac{\sigma_3 x_5}{\lambda_3 + x_5} \\
 223 \quad a_{54} &:= 0 \\
 224 \quad a_{55} &:= -\frac{\sigma_0 \lambda_0}{(\lambda_0 + x_5)^2} - \frac{\sigma_2 \lambda_2 x_2}{(\lambda_2 + x_5)^2} - \frac{\sigma_3 \lambda_3 x_3}{(\lambda_3 + x_5)^2}
 \end{aligned} \quad (4.2)$$

229 The Jacobian matrices for the steady states  $E_3, E_4, E_5$  are respectively listed as follows:

$$\begin{aligned}
& J\{E_1[0, 0, 0, 0, x_5]\} = \\
230 \quad & \begin{bmatrix} -k_1 & 0 & 0 & 0 & 0 \\ 0 & -k_2 - \frac{\xi_2 \sigma_2 x_5}{\lambda_2 + x_5} & -\beta_1 & 0 & 0 \\ 0 & 0 & \beta_3 - k_3 - \frac{\xi_3 \sigma_3 x_5}{\lambda_3 + x_5} & 0 & -\frac{\xi_3 \sigma_3 \lambda_3}{(\lambda_3 + x_5)^2} \\ 0 & 0 & 0 & -k_4 & 0 \\ 0 & -\frac{\sigma_2 x_5}{\lambda_2 + x_5} & -\frac{\sigma_3 x_5}{\lambda_3 + x_5} & 0 & -\frac{\sigma_0 \lambda_0}{(\lambda_0 + x_5)^2} \end{bmatrix} \quad (4.3)
\end{aligned}$$

$$\begin{aligned}
& J\{E_2[0, 0, 0, x_4, x_5]\} = \\
231 \quad & \begin{bmatrix} -k_1 & 0 & 0 & 0 & 0 \\ 0 & -k_2 - K_1 x_4 - \frac{\xi_2 \sigma_2 x_5}{\lambda_2 + x_5} & -\beta_1 & 0 & 0 \\ 0 & 0 & \beta_3 - \eta_1 x_4 - k_3 - \frac{\xi_3 \sigma_3 x_5}{\lambda_3 + x_5} & 0 & -\frac{\xi_3 \sigma_3 \lambda_3}{(\lambda_3 + x_5)^2} \\ a_4 x_4 & -K_2 x_4 & -\eta_2 x_4 & -k_4 & 0 \\ 0 & -\frac{\sigma_2 x_5}{\lambda_2 + x_5} & -\frac{\sigma_3 x_5}{\lambda_3 + x_5} & 0 & -\frac{\sigma_0 \lambda_0}{(\lambda_0 + x_5)^2} \end{bmatrix} \quad (4.4)
\end{aligned}$$

$$\begin{aligned}
& J\{E_3[x_1, 0, 0, x_4, x_5]\} = \\
232 \quad & \begin{bmatrix} a_1 x_1 (2 - b_1 x_1) e^{-b_1 x_1} - k_1 & -q_1 x_1 & -\alpha_1 x_1 & 0 & 0 \\ 0 & a_2 x_1 e^{-b_2 x_1} - q_2 x_1 - k_2 - K_1 x_4 - \frac{\xi_2 \sigma_2 x_5}{\lambda_2 + x_5} & \alpha_2 x_1 - \beta_1 & 0 & 0 \\ 0 & 0 & \beta_3 - \alpha_3 x_1 - \eta_1 x_4 - k_3 - \frac{\xi_3 \sigma_3 x_5}{\lambda_3 + x_5} & 0 & -\frac{\xi_3 \sigma_3 \lambda_3}{(\lambda_3 + x_5)^2} \\ a_4 x_4 (1 - b_4 x_1) e^{-b_4 x_1} & -K_2 x_4 & -\eta_2 x_4 & a_4 x_1 e^{-b_4 x_1} - k_4 & 0 \\ 0 & -\frac{\sigma_2 x_5}{\lambda_2 + x_5} & -\frac{\sigma_3 x_5}{\lambda_3 + x_5} & 0 & -\frac{\sigma_0 \lambda_0}{(\lambda_0 + x_5)^2} \end{bmatrix} \quad (4.5)
\end{aligned}$$

$$\begin{aligned}
& J\{E_4[x_1, 0, 0, 0, x_5]\} = \\
233 \quad & \\
234 \quad & \begin{bmatrix} a_1 x_1 (2 - b_1 x_1) e^{-b_1 x_1} - k_1 & -q_1 x_1 & -\alpha_1 x_1 & 0 & 0 \\ 0 & a_2 x_1 e^{-b_2 x_1} - q_2 x_1 - k_2 - \frac{\xi_2 \sigma_2 x_5}{\lambda_2 + x_5} & \alpha_2 x_1 - \beta_1 & 0 & 0 \\ 0 & 0 & \beta_3 - \alpha_3 x_1 - k_3 - \frac{\xi_3 \sigma_3 x_5}{\lambda_3 + x_5} & 0 & -\frac{\xi_3 \sigma_3 \lambda_3}{(\lambda_3 + x_5)^2} \\ 0 & 0 & 0 & a_4 x_1 e^{-b_4 x_1} - k_4 & 0 \\ 0 & -\frac{\sigma_2 x_5}{\lambda_2 + x_5} & -\frac{\sigma_3 x_5}{\lambda_3 + x_5} & 0 & -\frac{\sigma_0 \lambda_0}{(\lambda_0 + x_5)^2} \end{bmatrix} \quad (4.6)
\end{aligned}$$

$$\begin{aligned}
& J\{E_5[0, x_2, x_3, 0, x_5]\} = \\
235 \quad & \begin{bmatrix} -q_1 x_2 - k_1 - \alpha_1 x_3 & 0 & 0 & 0 & 0 \\ a_2 x_2 - q_2 x_2 & -k_2 - \frac{\xi_2 \sigma_2 x_5}{\lambda_2 + x_5} & -\beta_1 & -K_1 x_2 & -\frac{\xi_2 \lambda_2 \sigma_2 x_2}{(\lambda_2 + x_5)^2} \\ -\alpha_3 x_3 & \beta_2 x_3 & \beta_2 x_2 + \beta_3 - k_3 - \frac{\xi_3 \sigma_3 x_5}{\lambda_3 + x_5} & -\eta_1 x_3 & -\frac{\xi_3 \sigma_3 \lambda_3}{(\lambda_3 + x_5)^2} \\ 0 & 0 & 0 & -K_2 x_2 - \eta_2 x_3 - k_4 & 0 \\ 0 & -\frac{\sigma_2 x_5}{\lambda_2 + x_5} & -\frac{\sigma_3 x_5}{\lambda_3 + x_5} & 0 & -\frac{\sigma_0 \lambda_0}{(\lambda_0 + x_5)^2} - \frac{\sigma_2 \lambda_2 x_2}{(\lambda_2 + x_5)^2} - \frac{\sigma_3 \lambda_3 x_3}{(\lambda_3 + x_5)^2} \end{bmatrix}
\end{aligned}$$



(4.7)

$$\begin{aligned}
& J\{E_7[0, x_2, 0, 0, x_5]\} = \\
237 \quad & \begin{bmatrix} -q_1x_2 - k_1 & 0 & 0 & 0 & 0 \\ a_2x_2 - q_2x_2 & -k_2 - \frac{\xi_2\sigma_2x_5}{\lambda_2 + x_5} & -\beta_1 & -K_1x_2 & -\frac{\xi_2\lambda_2\sigma_2x_2}{(\lambda_2 + x_5)^2} \\ 0 & 0 & \beta_2x_2 + \beta_3 - k_3 - \frac{\xi_3\sigma_3x_5}{\lambda_3 + x_5} & 0 & -\frac{\xi_3\sigma_3\lambda_3}{(\lambda_3 + x_5)^2} \\ 0 & 0 & 0 & -K_2x_2 - k_4 & 0 \\ 0 & -\frac{\sigma_2x_5}{\lambda_2 + x_5} & -\frac{\sigma_3x_5}{\lambda_3 + x_5} & 0 & -\frac{\sigma_0\lambda_0}{(\lambda_0 + x_5)^2} - \frac{\sigma_2\lambda_2x_2}{(\lambda_2 + x_5)^2} \end{bmatrix}
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
& J\{E_9[x_1, x_2, 0, x_4, x_5]\} = \\
238 \quad & \begin{bmatrix} a_1x_1(2 - b_1x_1)e^{-b_1x_1} + q_1x_2 - k_1 & -q_1x_1 & -\alpha_1x_1 & 0 & 0 \\ a_2x_2(1 - b_2x_1)e^{-b_2x_1} + q_2x_2 & a_2x_1e^{-b_2x_1} - q_2x_1 - k_2 - K_1x_4 - \frac{\xi_2\sigma_2x_5}{\lambda_2 + x_5} & \alpha_2x_1 - \beta_1 & -K_1x_2 & -\frac{\xi_2\sigma_2\lambda_2x_2}{(\lambda_2 + x_5)^2} \\ 0 & 0 & \beta_2x_2 + \beta_3 - \alpha_3x_1 - \eta_1x_4 - k_3 - \frac{\xi_3\sigma_3x_5}{\lambda_3 + x_5} & 0 & -\frac{\xi_3\sigma_3\lambda_3}{(\lambda_3 + x_5)^2} \\ a_4x_4(1 - b_4x_1)e^{-b_4x_1} & -K_2x_4 & -\eta_2x_4 & a_4x_1e^{-b_4x_1} + K_2x_2 - k_4 & 0 \\ 0 & -\frac{\sigma_2x_5}{\lambda_2 + x_5} & -\frac{\sigma_3x_5}{\lambda_3 + x_5} & 0 & -\frac{\sigma_0\lambda_0}{(\lambda_0 + x_5)^2} - \frac{\sigma_2\lambda_2x_2}{(\lambda_2 + x_5)^2} \end{bmatrix}
\end{aligned} \tag{4.9}$$

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## 242 5. Necessary Criteria for Various Therapeutic Outcomes of AIDS during HAART

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244 In this section, the necessary mathematical criteria for all therapeutic outcomes during HAART are computed and  
 245 presented in the form of theorems.

246 **Theorem 5.1.** Suppose

$$\begin{aligned}
247 \quad & \text{(i)} \quad \begin{cases} S_1 + a_1\hat{x}_1^2e^{-b_1\hat{x}_1} - k_1\hat{x}_1 - e_{10} = 0 \\ S_2 - e_{20} = 0 \\ S_3 - e_{30} = 0 \\ S_4 + a_4\hat{x}_1\hat{x}_4e^{-b_4\hat{x}_1} - k_4\hat{x}_4 - e_{40} = 0 \\ D - \frac{\sigma_0\hat{x}_5}{\lambda_0 + \hat{x}_5} - k_5\hat{x}_5 = 0 \end{cases} \tag{5.1}
\end{aligned}$$

$$\begin{aligned}
248 \quad & \text{(ii)} \quad \begin{cases} a_1x_1(2 - b_1x_1)e^{-b_1x_1} - k_1 < 0 \\ a_2x_1e^{-b_2x_1} - q_2x_1 - k_2 - \frac{\xi_2\sigma_2x_5}{\lambda_2 + x_5} < 0 \\ \beta_3 - \alpha_3x_1 - k_3 - \frac{\xi_3\sigma_3x_5}{\lambda_3 + x_5} < 0 \\ a_4x_1e^{-b_4x_1} - k_4 < 0 \end{cases} \tag{5.2}
\end{aligned}$$

249 The HAART therapeutic outcome  $E_4[x_1, 0, 0, 0, x_5]$  exists and it is a local attractor.

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**Proof.** Consider the model equations (3.1). Then criterion (5.1) is a necessary condition for the existence of  $E_4[x_1, 0, 0, 0, x_5]$ . Now the Jacobian matrix of linearization of the model equations in the neighborhood of  $E_4[x_1, 0, 0, 0, x_5]$  is such that the eigenvalues are given by the following expressions.

$$\begin{cases} \lambda_1 = a_1 x_1 (2 - b_1 x_1) e^{-b_1 x_1} - k_1 \\ \lambda_2 = a_2 x_1 e^{-b_2 x_1} - q_2 x_1 - k_2 - \frac{\xi_2 \sigma_2 x_5}{\lambda_2 + x_5} \\ \lambda_3 = \beta_3 - \alpha_3 x_1 - k_3 - \frac{\xi_3 \sigma_3 x_5}{\lambda_3 + x_5} \\ \lambda_4 = a_4 x_1 e^{-b_4 x_1} - k_4 \\ \lambda_5 = -\frac{\sigma_0 \lambda_0}{(\lambda_0 + x_5)^2} \end{cases} \quad (5.3)$$

263 The eigenvalues have negative real parts when criterion (4.2) holds. Thus, two criteria (5.1) and (5.2) constitute the necessary  
264 conditions for the local existence of  $E_4[x_1, 0, 0, 0, x_5]$ . In particular, the principles of linearized stability of dynamical systems  
265 can be used to imply that the rest point  $E_4[x_1, 0, 0, 0, x_5]$  is locally asymptotically stable and hence a local attractor.  $\square$

266 **Clinical Implication 5.1.** The criteria (5.1) and (5.2) guarantee a temporary cure for the AIDS patient. There will be a finite  
267 time interval during which the HIV-1 virions will be annihilated from the patient's blood plasma. This will however be  
268 short-lived because the rest point  $E_4[x_1, 0, 0, 0, x_5]$  may become unstable and the criteria for temporal cure are violated. It is  
269 possible for therapeutic criteria to be derived to maintain the patient to be permanently free of AIDS, which we shall discuss in  
270 Theorem 5.5.

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272

**Theorem 5.2.** Suppose

$$(i) \begin{cases} S_1 - e_{10} = 0 \\ S_2 - k_2 \bar{x}_2 - \beta_1 \bar{x}_3 - e_{20} - \frac{\xi_2 \sigma_2 \bar{x}_2 \bar{x}_5}{\lambda_2 + x_5} = 0 \\ S_3 + \beta_2 \bar{x}_2 \bar{x}_3 + \beta_3 \bar{x}_3 - k_3 \bar{x}_3 - e_{30} - \frac{\xi_3 \sigma_3 \bar{x}_3 \bar{x}_5}{\lambda_3 + \bar{x}_5} = 0 \\ S_4 - e_{40} = 0 \\ D - \frac{\sigma_0 \bar{x}_5}{\lambda_0 + \bar{x}_5} - \frac{\sigma_2 \bar{x}_2 \bar{x}_5}{\lambda_2 + \bar{x}_5} - \frac{\sigma_3 \bar{x}_3 \bar{x}_5}{\lambda_3 + \bar{x}_5} - k_5 \bar{x}_5 = 0 \end{cases} \quad (5.4)$$

274 (ii) Let  $\sigma(J\{E_5[0, x_2, x_3, 0, x_5]\})$  be the eigen-spectrum of

$$J\{E_5[0, x_2, x_3, 0, x_5]\} := \begin{vmatrix} \bar{a}_{11} & \bar{a}_{12} & \bar{a}_{13} & \bar{a}_{14} & \bar{a}_{15} \\ \bar{a}_{21} & \bar{a}_{22} & \bar{a}_{23} & \bar{a}_{24} & \bar{a}_{25} \\ \bar{a}_{31} & \bar{a}_{32} & \bar{a}_{33} & \bar{a}_{34} & \bar{a}_{35} \\ \bar{a}_{41} & \bar{a}_{42} & \bar{a}_{43} & \bar{a}_{44} & \bar{a}_{45} \\ \bar{a}_{51} & \bar{a}_{52} & \bar{a}_{53} & \bar{a}_{54} & \bar{a}_{55} \end{vmatrix} \quad (5.5)$$

280 such that

$$281 \sigma(J\{E_5[0, x_2, x_3, 0, x_5]\}) = \{\lambda_i \mid \lambda^3 + \bar{a}_1 \lambda^2 + \bar{a}_2 \lambda + \bar{a}_3 = 0, i = 1, 2, 3\} \cup \{\lambda_4, \lambda_5\} \quad (5.6)$$

282 where

283

$$\begin{cases}
\bar{a}_1 = -\text{Trace}\{J\{E_5[0, x_2, x_3, 0, x_5]\}\} \\
\bar{a}_2 = \det \begin{vmatrix} \bar{a}_{33} & \bar{a}_{35} \\ \bar{a}_{53} & \bar{a}_{55} \end{vmatrix} + \det \begin{vmatrix} \bar{a}_{22} & \bar{a}_{25} \\ \bar{a}_{31} & \bar{a}_{33} \end{vmatrix} + \det \begin{vmatrix} \bar{a}_{22} & \bar{a}_{23} \\ \bar{a}_{32} & \bar{a}_{33} \end{vmatrix} \\
\bar{a}_3 = -\det J\{E_5[0, x_2, x_3, 0, x_5]\} \\
\lambda_4 = \bar{a}_{11} \\
\lambda_5 = \bar{a}_{44}
\end{cases} \quad (5.7)$$

284

Then the rest point  $E_5 = [0, x_2, x_3, 0, x_5]$  is locally asymptotically stable if

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$$\bar{a}_1 > 0, \bar{a}_3 > 0, \bar{a}_1 \bar{a}_2 > \bar{a}_3 \quad (5.8)$$

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**Proof.** The physiological outcome or rest point  $E_5$  exists if criterion (5.4) is satisfied. The Hartmann-Grobman theorem as applied in [10] in addition to the principles of linearized stability can be applied to the model equations in the neighborhood of  $E_5$ . The eigen-spectrum of the Jacobian matrix of the linearization is given by (5.6). Then the Routh-Hurwitz criterion [10] can be applied to (5.6) with definitions stated by (5.7). Thus, Theorem (5.2) follows immediately.

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**Clinical Implication 5.2.** This theorem depicts the conditions for one of the worst scenarios during AIDS therapy using continuous infusion HAART. If the conditions (5.4) and (5.8) hold, then the HAART therapy annihilated all the uninfected  $CD_4^+$  T cells, and HIV-1 specific cytotoxic  $CD_8^+$  T cells. Consequently, the immune system of the AIDS patient becomes incapacitated and the patient becomes the target of opportunistic infections. However, this situation may not last very long but could be fatal.

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**Theorem 5.3.** Suppose

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$$(i) \quad \begin{cases}
S_1 - e_{10} = 0 \\
S_2 - e_{20} = 0 \\
S_3 - e_{30} = 0 \\
S_4 - e_{40} = 0 \\
D - \frac{\sigma_0 \tilde{x}_5}{\lambda_0 + \tilde{x}_5} - k_5 \tilde{x}_5 = 0
\end{cases} \quad (5.9)$$

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The HAART therapeutic outcome  $E_1[0, 0, 0, 0, x_5]$  exists and it is a local attractor.

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**Proof.** The proof follows directly from the definition of the rest point and inspection of the eigen-spectrum of the Jacobian matrix of linearization of the model equations in the neighborhood of  $E_1$ .

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**Clinical Implication 5.3.** If the HAART protocol used for the AIDS patient is such that condition (5.9) is satisfied, then the patient experiences extreme immune system cytotoxicity in which the uninfected  $CD_4^+$  T cells, HIV-1 specific cytotoxic  $CD_8^+$  T cells, HIV-1 virions, and HIV-1 infected  $CD_4^+$  T cells are all decimated. This is not a therapeutically desirable clinical outcome, because the patient may become severely incapacitated. To avoid this scenario, the HAART therapy must be implemented in such a way that criterion (5.9) is violated.

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314 Let  $\sigma(J\{E_3[x_1, 0, 0, x_4, x_5]\})$  be the eigen-spectrum for the Jacobian matrix  $(J\{E_3[x_1, 0, 0, x_4, x_5]\})$  such that  
 315  $\sigma(J\{E_3[x_1, 0, 0, x_4, x_5]\}) = \{\lambda_k \mid \det |J(E_3) - \lambda I| = 0, k = 1, 2, 3, 4, 5\}$

316  
 317 **Theorem 5.4.** Suppose

$$318 \quad (i) \quad \begin{cases} S_1 - e_{10} = k_1 \bar{x}_1 - a_1 \bar{x}_1^2 e^{-b_1 \bar{x}_1} \\ S_4 - e_{40} = k_4 \bar{x}_4 - a_4 \bar{x}_1 \bar{x}_4 e^{-b_4 \bar{x}_1} \\ D = \frac{\sigma_0 \bar{x}_5}{\lambda_0 + \bar{x}_5} \end{cases} \quad (5.10)$$

322 (ii) The eigenvalues of  $(J\{E_3[x_1, 0, 0, x_4, x_5]\})$ , represented by  $\lambda_k = \alpha_k + i\beta_k$ , are such that the  $Re(\lambda_k) < 0$ .  
 323 Then the rest point of E3 exists and is a local attractor.  
 324

325 **Proof.** The condition (i) guarantees the local existence of E3. The principles of linearized stability can be used to show that  
 326 condition (ii) ensures that E3 is locally asymptotically stable and hence is a local attractor.  $\square$

327 Next, the interior rest point E10 is considered. This describes a situation in which the patient survives for a long time  
 328 with AIDS. It is possible during HAART that, the HIV-1 virions cannot be annihilated but instead the system persists in a  
 329 chronic configuration in which the blood plasma levels of the HIV-1 virions co-exist with the other clinically measurable  
 330 parameters such as uninfected CD4+ T cells, the HIV-1 infected CD4+ T cells, HIV-1 specific CD8+ T cells, and HAART drug  
 331 molecules. The criteria for existence of E10 are listed in the following theorem.

332 **Theorem 5.5.** Let

$$333 \quad (i) \quad \begin{cases} m_i = \inf_{t \in [t_0, T]} \{a_i \bar{x}_1 \bar{x}_i e^{-b_i \bar{x}_1}\} \quad \text{where } i = \{1, 2, 4\} \\ L_1 = \sup_{t \in [t_0, T]} \{\alpha_1 \bar{x}_3 + q_1 \bar{x}_2 + k_1\} \\ L_2 = \sup_{t \in [t_0, T]} \left\{ q_2 \bar{x}_1 + k_2 + K_1 \bar{x}_4 + \frac{\xi_2 \sigma_2 \bar{x}_2 \bar{x}_5}{\lambda_2 + \bar{x}_5} \right\} \\ U_1 = \sup_{t \in [t_0, T]} \{\beta_1 \bar{x}_3\} \\ L_3 = \sup_{t \in [t_0, T]} \left\{ \alpha_3 \bar{x}_1 + \eta_1 \bar{x}_4 + k_3 + \frac{\xi_3 \sigma_3}{\lambda_3 + \bar{x}_5} \right\} \\ m_3 = \inf_{t \in [t_0, T]} \{\beta_2 \bar{x}_2 \bar{x}_3 + \beta_3 \bar{x}_3\} \\ L_4 = \inf_{t \in [t_0, T]} \{K_2 \bar{x}_2 + \eta_2 \bar{x}_3 + k_4\} \\ L_5 = \inf_{t \in [t_0, T]} \left\{ \frac{\sigma_0}{\lambda_0 + \bar{x}_5} + \frac{\sigma_2 \bar{x}_2}{\lambda_2 + \bar{x}_5} + \frac{\sigma_3 \bar{x}_3}{\lambda_3 + \bar{x}_5} \right\} \end{cases} \quad (5.11)$$

$$353 \quad (ii) \quad \begin{cases} \liminf x_1 \geq \frac{S_1 + m_1 - e_{10}}{L_1} > 0 \\ \liminf x_2 \geq \frac{S_2 + m_2 - U_1 - e_{20}}{L_2} > 0 \\ \liminf x_3 \geq \frac{S_3 + m_3 - e_{30}}{L_3} > 0 \\ \liminf x_4 \geq \frac{S_4 + m_4 - e_{40}}{L_4} > 0 \\ \liminf x_5 \geq \frac{D}{L_5} > 0 \end{cases} \quad (5.12)$$

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361 Then the system  $[x_1, x_2, x_3, x_4, x_5]$  will persist and the HAART therapeutic outcome  $E_{10}=[\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5]$  exists.

362 **Proof.** For a proof refer to Nani and Jin[10]. □

363

364 **Clinical Implication 5.4.** If uninfected  $CD_4^+$  T cells, infected  $CD_4^+$  T cells, the HIV-1 viral mRNA copies in the blood plasma,  
365 HIV-1 specific cytotoxic  $CD_8^+$  T cells, and the HAART drug concentration are within certain thresholds, then the AIDS patient  
366 will live with AIDS. If any of the conditions listed in (5.11) are violated, the patient will experience unpredictable therapeutic  
367 outcomes including full-blown AIDS or spontaneous elimination of HIV-1 virions.  
368

## 369 6. Sufficient Criteria for Permanent Cure of AIDS during HAART

370 In this section, the sufficient criteria for permanent annihilation of HIV-1 virions during HAART will be derived. The  
371 clinically desirable physiological outcomes are  $E_4[x_1, 0, 0, 0, x_5]$  and  $E_3=[x_1, 0, 0, x_4, x_5]$ . It must be recalled that  $E_4$   
372 corresponds to the physiological outcome in which HIV-1 infected  $CD_4^+$  T cells, the HIV-1 virions in the blood plasma and  
373 HIV-1 specific  $CD_8^+$  T cells are eliminated in the blood plasma of the AIDS patient. The sufficient criteria for  $E_4$  will be  
374 discussed first.

375 The rest point  $E_4$  will be analyzed for global asymptotical stability in the space  $R_+^{x_1, x_5} = [x_1, x_5 | x_1 \geq 0, x_5 \geq 0]$

376 The model equations (3.1) correspondingly reduce to the following:

$$\begin{cases} \dot{x}_1 = S_1 + a_1 x_1^2 e^{-b_1 x_1} - k_1 x_1 - e_{10} \\ \dot{x}_5 = D - \frac{\sigma_0 x_5}{\lambda_0 + x_5} \\ x_i(t_0) = x_{i0} \quad \text{for } i = \{1, 5\} \end{cases} \quad (6.1)$$

382 Consider the Liapunov functional:

$$\begin{aligned} 383 \quad V &:= \sum \frac{1}{2} \hat{c}_i (x_i - \hat{x}_i)^2 \\ 384 \quad &\text{where } i = \{1, 5\} \text{ and } \hat{c}_i \in R_+ = (0, \infty) \end{aligned} \quad (6.2)$$

386 The derivative of  $V$  along the solution curves of the model equations yields the result:

$$\begin{aligned} 387 \quad \dot{V} &= \hat{c}_1 (x_1 - \hat{x}_1) \dot{x}_1 + \hat{c}_5 (x_5 - \hat{x}_5) \dot{x}_5 \\ &= \hat{c}_1 (x_1 - \hat{x}_1) (S_1 + a_1 x_1^2 e^{-b_1 x_1} - k_1 x_1 - e_{10}) + \\ &\quad \hat{c}_5 (x_5 - \hat{x}_5) \left( D - \frac{\sigma_0 x_5}{\lambda_0 + x_5} \right) \end{aligned} \quad (6.3)$$

388 But at a steady state, the following equations hold.

$$\begin{cases} S_1 - e_{10} = k_1 \hat{x}_1 - a_1 \hat{x}_1^2 e^{-b_1 \hat{x}_1} \\ D = \frac{\sigma_0 \hat{x}_5}{\lambda_0 + \hat{x}_5} \end{cases} \quad (6.4)$$

392 Define the following Lebesgue measurable, functions which are of bounded variation:

$$\begin{aligned} 393 \quad G(x_1) &= a_1 x_1^2 e^{-b_1 x_1} \\ L(x_5) &= \frac{\sigma_0 x_5}{\lambda_0 + x_5} \end{aligned} \quad (6.5)$$

$$\begin{aligned}
394 \quad V^* &= \hat{c}_1(x_1 - \hat{x}_1)\dot{x}_1 + \hat{c}_5(x_5 - \hat{x}_5)\dot{x}_5 \\
&= \hat{c}_1(x_1 - \hat{x}_1)(-a_1\hat{x}_1^2 e^{-b_1x_1} + k_1\hat{x}_1 + a_1x_1^2 e^{-b_1x_1} - k_1x_1) + \hat{c}_5(x_5 - \hat{x}_5)\left(\frac{\sigma_0\hat{x}_5}{\lambda_0 + \hat{x}_5} - \frac{\sigma_0x_5}{\lambda_0 + x_5}\right) \\
&= \hat{c}_1(x_1 - \hat{x}_1)[G(x_1) - G(\hat{x}_1)] + \hat{c}_1(x_1 - \hat{x}_1)(k_1\hat{x}_1 - k_1x_1) \\
&\quad + \hat{c}_5(x_5 - \hat{x}_5)[L(\hat{x}_5) - L(x_5)]
\end{aligned} \tag{6.6}$$

$$395 \quad V^* = -\hat{c}_1k_1(x_1 - \hat{x}_1)^2 + \hat{c}_1(x_1 - \hat{x}_1)[G(x_1) - G(\hat{x}_1)] - \sigma_0\hat{c}_5(x_5 - \hat{x}_5)[L(x_5) - L(\hat{x}_5)] \tag{6.7}$$

Let

$$\begin{aligned}
397 \quad v_1 &= x_1 - \hat{x}_1 \\
398 \quad v_2 &= x_5 - \hat{x}_5
\end{aligned} \tag{6.8}$$

and

$$400 \quad X = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}_+^2 \tag{6.9}$$

and define  $A = \{a_{ij}\} \in M_{2 \times 2}(\mathbb{R})$  such that

$$405 \quad A = \begin{bmatrix} a_{11} & \frac{1}{2}a_{12} \\ \frac{1}{2}a_{21} & a_{22} \end{bmatrix}$$

then

$$\begin{aligned}
408 \quad V^* &:= a_{11}v_1^2 + \frac{1}{2}a_{12}v_1v_2 + \frac{1}{2}a_{21}v_2v_1 + a_{22}v_2^2 \\
409 \quad &= X^T A X
\end{aligned} \tag{6.10}$$

410 Where  $X^T$  denotes the transpose of  $X$  and  $V^*$  is negative definite if the eigenvalues of  $A$  have negative real parts or  
411 satisfy the conditions of  $a_{11} < 0$  and  $\det(A) > 0$  [???].

412 In particular, the  $[a_{ij}]_{2 \times 2}$  are defined as follows:

$$\begin{cases}
413 \quad a_{11} := -[\hat{c}_1k_1 - \hat{c}_1\left(\frac{G(x_1) - G(\hat{x}_1)}{x_1 - \hat{x}_1}\right)] \\
414 \quad a_{12} = a_{21} = 0 \\
415 \quad a_{22} := -\hat{c}_5\left[\frac{L(x_5) - L(\hat{x}_5)}{x_5 - \hat{x}_5}\right]
\end{cases} \tag{6.11}$$

418 As the flow dynamics approaches the steady state  $E_4[x_1, 0, 0, 0, x_5]$ , the following conditions hold:

$$\begin{aligned}
419 \quad a_{11} &\rightarrow -[\hat{c}_1k_1 - G'_1(\hat{x}_1)] \\
420 \quad a_{22} &\rightarrow -\hat{c}_5L'(\hat{x}_5)
\end{aligned} \tag{6.12}$$

422 where

$$\begin{aligned}
423 \quad G'(\hat{x}_1) &= a_1\hat{x}_1 e^{-b_1\hat{x}_1} (2 - b_1\hat{x}_1) \\
424 \quad L'(\hat{x}_5) &= \frac{\sigma_0\lambda_0}{(\lambda_0 + \hat{x}_5)^2} > 0
\end{aligned}$$

426 In particular,

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$$G'(\hat{x}_1) = \begin{cases} > 0 & \text{if } \hat{x}_1 < \frac{2}{b_1} \\ = 0 & \text{if } \hat{x}_1 = \frac{2}{b_1} \\ < 0 & \text{if } \hat{x}_1 > \frac{2}{b_1} \end{cases}$$

The sufficient criteria for the global asymptotic stability of  $E_4$  are specified in the following theorem.

**Theorem 6.1** Let

- (i) Conditions (5.1) and (5.2) hold,
- (ii) Suppose  $\hat{x}_1 < \frac{2}{b_1}$  holds,

then the rest point  $E_4$  is a global attractor such that the HIV-1 virions, HIV-1 infected CD4+ T cells, HIV-1 specific CD8+ T cells are annihilated during HAART.

**Proof.** Condition (i) guarantees the local existence and local asymptotic stability of  $E_4$ . If condition (ii) holds, then eigenvalues of the matrix  $A$  has negative real parts, and consequently,  $\hat{V}^*$  is negative definite. Hence, the theorem holds.  $\square$

**Clinical Implication 6.1**

- (i) The criterion  $\hat{x}_1 < \frac{2}{b_1}$  is a sufficient condition for annihilation of HIV-1 virions of the blood plasma of AIDS patient during HAART.
- (ii) The function  $G(\hat{x}_1)$  is a measure of the rate at which the CD4+ T cells ( $x_1$ ) are activated by interleukin-2 (IL-2) in an autocrine process when the physiological flow during HAART tends to  $E_4$ . It is possible to attach a biochemical measure to coefficient  $a_1$  and  $b_1$  of  $G(\hat{x}_1)$ . This can be done as follows:

$$\begin{aligned} G(\hat{x}_1) &= a_1 \hat{x}_1^2 e^{-b_1 \hat{x}_1} = \frac{a_1 \hat{x}_1^2}{e^{b_1 \hat{x}_1}} = \frac{a_1 \hat{x}_1 \hat{x}_1}{1 + b_1 \hat{x}_1 + \frac{(b_1 \hat{x}_1)^2}{2!} + \dots} \approx \frac{a_1 \hat{x}_1 \hat{x}_1}{1 + b_1 \hat{x}_1} = \frac{\frac{a_1}{b_1} \hat{x}_1 \hat{x}_1}{\frac{1}{b_1} + \hat{x}_1} \\ &= \frac{V_{\max} \hat{x}_1 \hat{x}_1}{K_m + \hat{x}_1} \end{aligned}$$

Where  $K_m = \frac{1}{b_1}$  is approximately equal to the Michaelis-Menten constant of the CD4+ T cells activation reaction by interleukin-2, and  $V_{\max}$  represents the maximal velocity of the CD4+ T cells IL-2 activation reaction.

- (iii) Note that  $k_1$  is the rate of degradation and decay of the CD4+ T cells during AIDS. Thus the theorem implies that if the rate of degradation of CD4+ T cells is greater than the rate of IL-2 activation of T cells and if in addition, the number of the CD4+ T cells are such that  $\hat{x}_1 < 2K_m$ , then the patient will be cured of AIDS.

Next, the other clinical desirable rest point  $E_3 = [x_1, 0, 0, x_4, x_5]$  is analyzed. The aim is to find the sufficient criteria under which the HIV-1 virions in the blood plasma, the HIV-1 infected CD4+ T cells are annihilated but the HIV-1 specific CD8+ T cells will persist as memory T cells as well as the CD4+ T cells and some drug residues will remain in the AIDS patient. It is expected that the CD4+ T cells will eventually repopulate to their carrying capacity whereas the drug residues will ultimately dissipate. The analysis will be similar to the preceding one for  $E_4$ .

The restriction of the model equations to the space  $R_+^{x_1, x_4, x_5} = [x_1, x_4, x_5 \mid x_1 \geq 0, x_4 \geq 0, x_5 \geq 0]$  leads to the following equations [nani and jin].

$$\begin{cases}
\dot{x}_1 = S_1 + a_1 x_1^2 e^{-b_1 x_1} - k_1 x_1 - e_{10} \\
\dot{x}_4 = S_4 + a_4 x_1 x_4 e^{-b_4 x_1} - k_4 x_4 - e_{40} \\
\dot{x}_5 = D - \frac{\sigma_0 x_5}{\lambda_0 + x_5} \\
x_i(t_0) = x_{i0} \quad \text{for } i = \{1, 4, 5\}
\end{cases} \quad (6.13)$$

Consider the Liapnnov functional:

$$V := \sum \frac{1}{2} \bar{c}_i (x_i - \bar{x}_i)^2 \quad (6.14)$$

where  $i = \{1, 4, 5\}$  and  $c_i \in R_+ = (0, \infty)$

The derivative of V along the solution curves of the model equations yields the result:

$$\dot{V} = \bar{c}_1 (x_1 - \bar{x}_1) \dot{x}_1 + \bar{c}_4 (x_4 - \bar{x}_4) \dot{x}_4 + \bar{c}_5 (x_5 - \bar{x}_5) \dot{x}_5$$

Thus a steady state,  $\dot{x}_i = 0$  and the following equations hold.

$$\begin{cases}
S_1 - e_{10} = k_1 \bar{x}_1 - a_1 \bar{x}_1^2 e^{-b_1 \bar{x}_1} \\
S_4 - e_{40} = k_4 \bar{x}_4 - a_4 \bar{x}_1 \bar{x}_4 e^{-b_4 \bar{x}_1} \\
D = \frac{\sigma_0 \bar{x}_5}{\lambda_0 + \bar{x}_5}
\end{cases} \quad (6.15)$$

Thus

$$\begin{aligned}
\dot{V} &= \bar{c}_1 k_1 (x_1 - \bar{x}_1) (\bar{x}_1 - x_1) \\
&\quad + \bar{c}_1 (x_1 - \bar{x}_1) [G(\bar{x}_1) - G(x_1)] \\
&\quad + \bar{c}_4 k_4 (x_4 - \bar{x}_4) (\bar{x}_4 - x_4) \\
&\quad + \bar{c}_4 (x_4 - \bar{x}_4) [F(\bar{x}_1, \bar{x}_4) - F(x_1, x_4)] \\
&\quad + \bar{c}_5 \sigma_0 [L(\bar{x}_5) - L(x_5)]
\end{aligned}$$

where

$$G(x_1) = a_1 x_1^2 e^{-b_1 x_1} \quad (6.16)$$

$$F(x_1, x_4) = a_4 x_1 x_4 e^{-b_4 x_1}$$

$$L(x_5) = \frac{\sigma_0 x_5}{\lambda_0 + x_5}$$

The functions  $G, F, L$  are continuous, Lebesgue measurable, and of bounded variations.

$$\text{Let } \dot{V} = X^T C X \quad (6.17)$$

where

$$\text{and } X = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in R_+^3$$

$$\begin{aligned}
u_1 &= x_1 - \bar{x}_1 \\
u_2 &= x_4 - \bar{x}_4 \\
u_3 &= x_5 - \bar{x}_5
\end{aligned} \quad (6.18)$$



498 such that  $X^T$  denotes the transpose of  $X$ . Define  $C \in M_{3 \times 3}(R)$  such that

$$499$$

$$500 \quad C = \begin{bmatrix} c_{11} & \frac{1}{2}c_{12} & \frac{1}{2}c_{13} \\ \frac{1}{2}c_{12} & c_{22} & c_{23} \\ \frac{1}{2}c_{13} & \frac{1}{2}c_{23} & c_{33} \end{bmatrix} \quad (6.19)$$

503 Now

$$504 \quad V^* := c_{11}u_1^2 + \frac{1}{2}c_{12}u_1u_2 + \frac{1}{2}c_{13}u_1u_3$$

$$505 \quad + \frac{1}{2}c_{12}u_2u_1 + c_{22}u_2^2 + \frac{1}{2}c_{23}u_2u_3 \quad (6.20)$$

$$506 \quad + \frac{1}{2}c_{13}u_3u_1 + \frac{1}{2}c_{23}u_3u_2 + c_{33}u_3^2$$

508 Where the  $[c_{ij}]_{3 \times 3}$  are defined as follows:

$$509 \quad \begin{cases} c_{11} := -[\bar{c}_1 k_1 - c_1 \left( \frac{G(x_1) - G(\bar{x}_1)}{x_1 - \bar{x}_1} \right)] \\ c_{12} := -c_4 \left[ \frac{F(x_1, x_4) - F(\bar{x}_1, \bar{x}_4)}{x_1 - \bar{x}_1} \right] = c_{21} \\ c_{13} = c_{31} = 0 \\ c_{22} = -\bar{c}_4 k_4 \\ c_{23} = c_{32} = 0 \\ c_{33} := -c_5 \left[ \frac{L(x_5) - L(\bar{x}_5)}{x_5 - \bar{x}_5} \right] \end{cases} \quad (6.21)$$

513 As the flow associated with the model equations approaches  $E_3 = [\bar{x}_1, 0, 0, \bar{x}_4, \bar{x}_5]$ , the matrix entries  $[c_{ij}]_{3 \times 3}$  have the

514 following form:

$$515 \quad c_{11} \rightarrow -[\bar{c}_1 k_1 - \bar{c}_1 G'(\bar{x}_1)]$$

$$516 \quad c_{12} \rightarrow -c_4 [F_{x_1}(\bar{x}_1, \bar{x}_4)]$$

$$517 \quad c_{22} = -\bar{c}_4 k_4 \quad (6.22)$$

$$518 \quad c_{33} \rightarrow -\bar{c}_5 L'(\bar{x}_5)$$

519 But it can be shown that

$$520 \quad F_{x_1}(\bar{x}_1, \bar{x}_4) = a_4 \bar{x}_4 e^{-b_4 \bar{x}_1} (1 - \bar{x}_1 b_4)$$

$$521 \quad G'(\bar{x}_1) = a_1 \bar{x}_1 e^{-b_1 \bar{x}_1} (2 - \bar{x}_1 b_1) \quad (6.23)$$

$$522 \quad L'(\bar{x}_5) = \frac{\sigma_0 \lambda_0}{(\lambda_0 + \bar{x}_5)^2} > 0$$

525 In particular, (cf. [nani and jin])

$$526 \quad F_{x_1}(\bar{x}_1, \bar{x}_4) = \begin{cases} > 0 \text{ if } \bar{x}_1 < \frac{1}{b_4} \\ = 0 \text{ if } \bar{x}_1 = \frac{1}{b_4} \\ < 0 \text{ if } \bar{x}_1 > \frac{1}{b_4} \end{cases} \quad (6.24)$$

530 Similarly,

$$531 \quad G'(\bar{x}_1) = \begin{cases} > 0 \text{ if } \bar{x}_1 < \frac{2}{b_1} \\ 532 \quad = 0 \text{ if } \bar{x}_1 = \frac{2}{b_1} \\ < 0 \text{ if } \bar{x}_1 > \frac{2}{b_1} \end{cases} \quad (6.25)$$

534 Using the specifications in (4.31), the matrix  $C$  has the form

$$535 \quad C = \begin{bmatrix} c_{11} & \frac{1}{2}c_{12} & 0 \\ 536 \quad \frac{1}{2}c_{12} & c_{22} & 0 \\ 537 \quad 0 & 0 & c_{33} \end{bmatrix} \quad (6.26)$$

The matrix  $C$  is negative definite if the following criteria hold:

$$539 \quad i. \quad C_1 = \det c_{11} < 0 \quad \text{or} \quad c_{11} < 0$$

$$540 \quad ii. \quad C_2 = \det \begin{vmatrix} c_{11} & \frac{1}{2}c_{12} \\ \frac{1}{2}c_{12} & c_{22} \end{vmatrix} > 0 \quad \text{or} \quad c_{11}c_{22} - \frac{1}{4}(c_{12})^2 > 0$$

$$541 \quad iii. \quad C_3 = \det \begin{vmatrix} c_{11} & \frac{1}{2}c_{12} & 0 \\ \frac{1}{2}c_{12} & c_{22} & 0 \\ 542 \quad 0 & 0 & c_{33} \end{vmatrix} < 0 \quad \text{or} \quad c_{33}[c_{11}c_{22} - \frac{1}{4}(c_{12})^2] < 0$$

546 **Theorem 6.2.** Suppose  $\bar{x}_1 = \frac{1}{b_4} < \frac{2}{b_1}$

547 Then the physiological steady state  $E_3 = [\bar{x}_1, 0, 0, \bar{x}_4, \bar{x}_5]$  is globally asymptotically stable and a global attractor.

548 **Proof.** The condition (6.27) *i.* holds when  $k_1 > G'(\bar{x}_1) > 0$ . But  $G'(\bar{x}_1) > 0$  when  $\bar{x}_1 < \frac{2}{b_1}$ . Observe that if  $\bar{x}_1 = \frac{1}{b_4}$ , then  
549  $c_{12} = 0$  and condition (6.27) *ii.* holds. Since  $c_{33} < 0$  and  $c_{12} = 0$ , the condition (4.37) also holds immediately. Hence the  
550 theorem follows.  $\square$

551 **Clinical Implications 6.2.**

552 (i) Theorem 4.6 gives the sufficient theoretical criteria for the HIV-1 infected CD4+ T cells and HIV-1 virions in the blood  
553 plasma to be eradicated during HAART. This theorem also provides the conditions under which HIV-1 specific CD8+ T cells  
554 will persist as memory cells after the HAART therapy.

555 (ii) It is possible to express the conditions of Theorem 4.6 in terms of clinically measurable biophysical parameters. By using  
556 Taylor expansions of  $e^{-b_1\bar{x}_1}$  and  $e^{-b_4\bar{x}_1}$ , the conditions  $\bar{x}_1 = \frac{1}{b_4} < \frac{2}{b_1}$  is approximately equivalent to the expression

$$557 \quad \bar{x}_1 = K_m^{CD8+} < 2K_m^{CD4+} \quad (6.28)$$

558 This can be interpreted to mean that HAART will cure the AIDS patient if the number density of uninfected CD4+ T cells is  
559 equal to the Michaelis-Menten constant of the IL-2 activation of the CD8+ T cells, which in turn has to be less than twice  
560 Michaelis-Menten constant of the IL-2 activation of the CD4+ T cells.

561 **Corollary 6.2.** Suppose  $\bar{x}_1 = \frac{1}{b_4} = \frac{2}{b_1}$

562 Then the physiological steady state  $E_3 = [\bar{x}_1, 0, 0, \bar{x}_4, \bar{x}_5]$  is a global attractor.

563 **Proof.** The conditions (4.37) *i, ii, iii* hold if the theorem hypothesis is satisfied. This makes the matrix  $C$  in (4.27) negative  
564 definite. Consequently, the steady state becomes global attractor.  $\square$

565 It is possible to combine Theorem 4.6 and Corollary 4.6 to obtain the combines criteria:

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$$\bar{x}_1 = \frac{1}{b_4} \leq \frac{2}{b_1}$$

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$$\bar{x}_1 \approx K_m^{CD8+} < 2K_m^{CD4+}$$

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574 **7. Simulation Results**

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In this section investigative computer simulations will be presented and discussed. These simulations are performed using clinically plausible hypothetical patient patho-physiological parametric configurations. The numerical estimates used are variations of estimates published in the references: [2,11,13,15]. Some techniques for estimating HIV-I dynamical parameters have been described by Ciupe et al. [2]. It must be emphasized that every AIDS patient has a unique situation and therefore these simulation results are hypothetical scenarios and depict non-equilibrium disease configurations but do not depict the equilibrium AIDS configurations represented by the necessary and sufficient theorems discussed in section 6. However the investigative computer simulations elucidate several dynamical aspects of HIV-1 AIDS dynamics which are based on the respective parametric configurations. In a future publication the equilibrium parametric configuration simulations will be presented. The x-axis of the simulation graphs is calibrated in months.

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584 **7.1 Simulation results for hypothetical AIDS patient #1**

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The AIDS patho-physiological parametric configuration of hypothetical patient #1 is denoted by  $P_1$  and exhibited in Table 1. A brief inspection of the data reveals explicitly that it is a non-equilibrium scenario. The simulation results show that the HAART protocol impairs HIV-1 dynamics in this patient leading to eradication of plasma HIV-1 viremia. The simulation results for patient #1 are exhibited in Figure 1. It can be observed that the HIV-1 infected CD4+ T cells are eradicated in this patient with the use of anti-AIDS pharmacotherapeutic drug protocols. In addition, patient #1 experiences HAART-induced immune system reconstitution as the uninfected CD4+ T cells and HIV-1 specific CD8+ T cells repopulate.

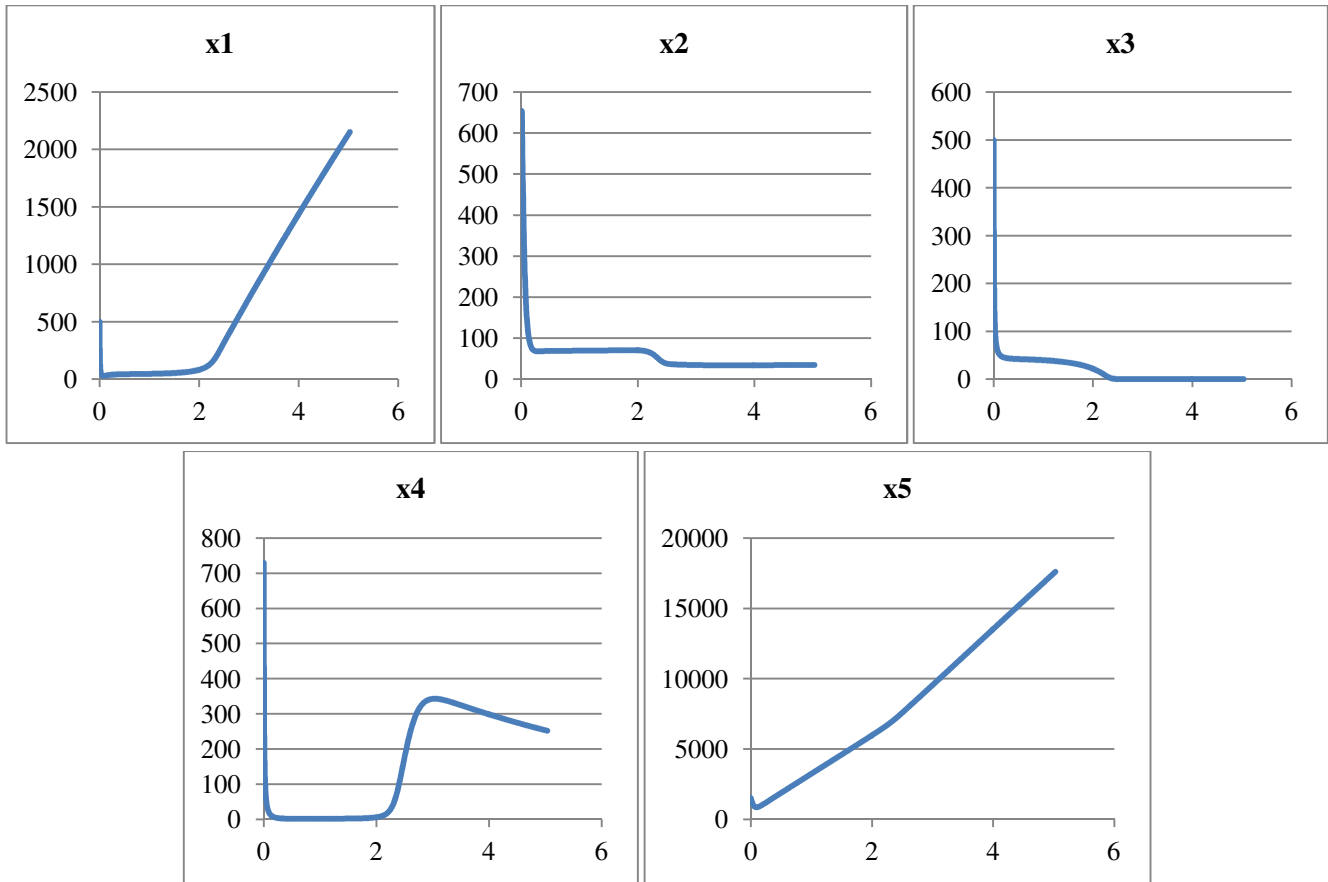
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**Table 1 Parametric Configuration  $P_1$**

TABLE I. Hypothetical AIDS Patient Parametric Configuration  $P_1$

$S_1 = 800 \text{ /day/}\mu\text{l}$ $a_1 = 0.15 \text{ /day/cell/}\mu\text{l}$ $b_1 = 0.019 \text{ /cell/}\mu\text{l}$ $\alpha_1 =$ $0.5 \text{ /day/virion/}\mu\text{l}$ $k_1 = 0.0005 \text{ /day/}\mu\text{l}$ $q_1 =$ $0.00045 \text{ /day/}\mu\text{l/cell}$ $e_{10} =$ $0.0025 \text{ cells/day/}\mu\text{l}$ $x_{10} = 500 \text{ cells/}\mu\text{l}$	$S_2 = 800 \text{ /day/}\mu\text{l}$ $a_2 = 0.03 \text{ /day/cell/}\mu\text{l}$ $b_2 = 0.004 \text{ /cell/}\mu\text{l}$ $\alpha_2 = 0.5 \text{ /day/virion/}\mu\text{l}$ $k_2 = 0.005 \text{ /day/}\mu\text{l}$ $q_2 = 0.00001 \text{ /day/}\mu\text{l/cell}$ $\beta_1 = 1.05$ $\text{virions/CD4}^+ \text{ /day}$ $K_1 = 0.0001 \text{ /day/}\mu\text{l}$ $e_{20} = 0.005 \text{ cells/day/}\mu\text{l}$ $\xi_2 = 0.85$ $x_{20} = 400 \text{ cells/}\mu\text{l}$	$S_3 = 10 \text{ /day/}\mu\text{l}$ $\beta_2 = 0.0015$ $\text{virions/CD4}^+ \text{ /day/}\mu\text{l}$ $\beta_3 = 1.05$ $\text{virions/CD4}^+ \text{ /day}$ $\alpha_3 = 0.027 \text{ /day/virion/}\mu\text{l}$ $k_3 = 0.0001 \text{ /day/}\mu\text{l}$ $e_{30} = 0.0001 \text{ /day}$ $\eta_1 = 0.25$ $\xi_3 = 0.001$ $x_{30} = 500 \text{ cells/}\mu\text{l}$	$S_4 = 10 \text{ /day/}\mu\text{l}$ $a_4 = 0.35 \text{ /day/cell/}\mu\text{l}$ $b_4 = 0.01 \text{ /cell/}\mu\text{l}$ $K_2 = 0.0024 \text{ /day/}\mu\text{l}$ $k_4 = 0.08 \text{ /day/}\mu\text{l}$ $e_{40} = 0.0002$ $\text{cells/day/}\mu\text{l}$ $\eta_2 = 0.45$ $x_{40} = 730 \text{ cells/}\mu\text{l}$	$D = 5000 \text{ units}$ $\sigma_0 = 0.5 \text{ mg/day}$ $\sigma_2 = 30 \text{ mg/day}$ $\sigma_3 = 5 \text{ mg/day}$ $\lambda_0 = 5 \text{ mg/L}$ $\lambda_2 = 15 \text{ mg/L}$ $\lambda_3 = 0.025 \text{ mg/L}$ $x_{50} = 1500 \text{ cells/}\mu\text{l}$ $k_5 = 0.0 \text{ /day}$ $\text{constant infusion}$
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**Figure 1 Simulation results using parametric configuration  $P_1$**

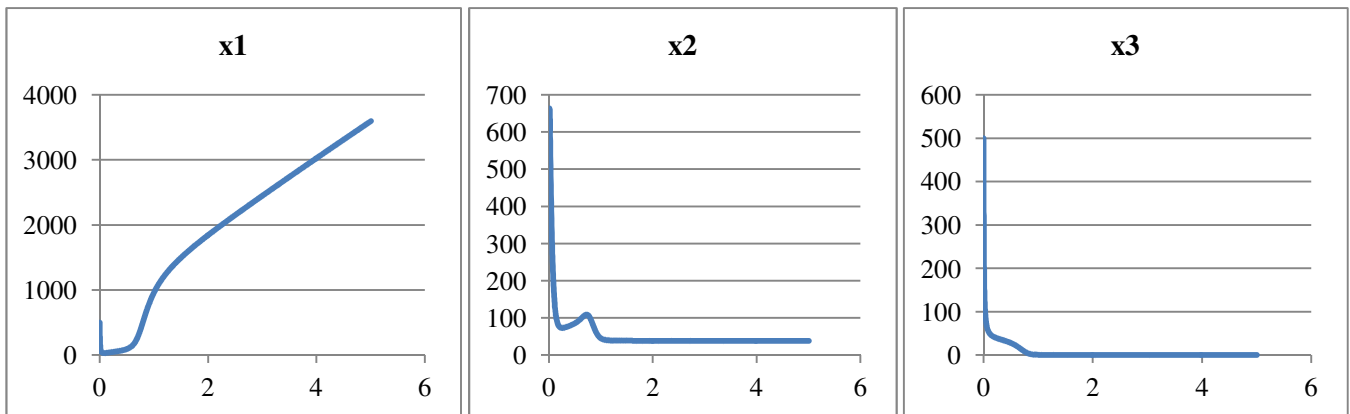
## 7.2 Simulation results for hypothetical AIDS patient #2

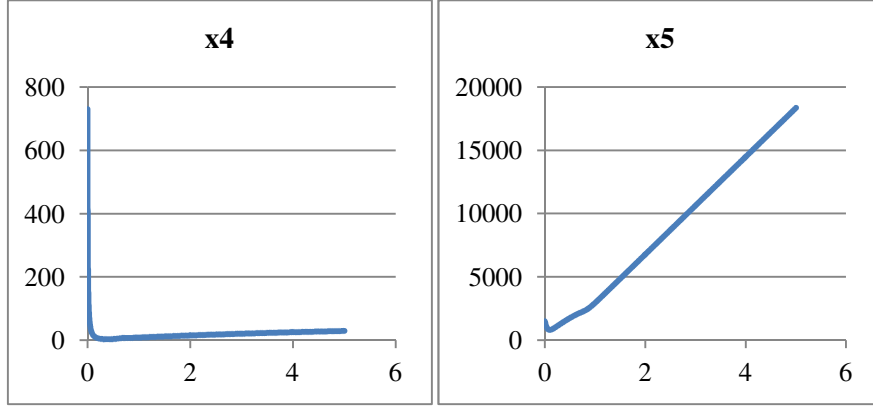
The hypothetical AIDS patient #2 is assigned the patho-physiological parameter configuration  $P_2$  as presented in Table 2. As in the previous simulation, the configuration  $P_2$  does not depict an equilibrium configuration. The simulation results are exhibited in Figure 2. It can be observed that this patient does have a clinically favorable prognosis under HAART treatment protocol. The AIDS patho-physiodynamics during the given HAART protocol is similar to that for patient #1, as the patient #2 also undergoes immune system reconstitution in which the CD4+ T cells repopulate. On the other hand, the proliferative activity of the HIV-1 specific CD8+ T cells appears to be down-regulated..

**Table 2 Parametric Configuration  $P_2$**

TABLE 2. Hypothetical AIDS Patient Parametric Configuration  $P_2$

$S_1 = 800$ /day/ $\mu$ l $a_1 = 0.15$ /day/cell/ $\mu$ l $b_1 = 0.005$ /cell/ $\mu$ l $\alpha_1 = 0.5$ /day/virion/ $\mu$ l $k_1 = 0.0005$ /day/ $\mu$ l $q_1 =$ $0.00045$ /day/ $\mu$ l/cell $e_{10} = 0.0025$ cells/day/ $\mu$ l $x_{10} = 500$ cells/ $\mu$ l	$S_2 = 800$ /day/ $\mu$ l $a_2 = 0.03$ /day/cell/ $\mu$ l $b_2 = 0.004$ /cell/ $\mu$ l $\alpha_2 = 0.5$ /day/virion/ $\mu$ l $k_2 = 0.005$ /day/ $\mu$ l $q_2 =$ $0.00001$ /day/ $\mu$ l/cell $\beta_1 = 1.05$ virions/CD4 <sup>+</sup> /day $K_1 = 0.0001$ /day/ $\mu$ l $e_{20} = 0.005$ cells/day/ $\mu$ l $\xi_2 = 0.85$ $x_{20} = 400$ cells/ $\mu$ l	$S_3 = 10$ /day/ $\mu$ l $\beta_2 = 0.0015$ virions/CD4 <sup>+</sup> /day/ $\mu$ l $\beta_3 = 1.05$ virions/CD4 <sup>+</sup> /day $\alpha_3 = 0.027$ /day/virion/ $\mu$ l $k_3 = 0.0001$ /day/ $\mu$ l $e_{30} = 0.0001$ /day $\eta_1 = 0.25$ $\xi_3 = 0.001$ $x_{30} = 500$ cells/ $\mu$ l	$S_4 = 10$ /day/ $\mu$ l $a_4 = 0.35$ /day/cell/ $\mu$ l $b_4 = 0.01$ /cell/ $\mu$ l $K_2 = 0.0024$ /day/ $\mu$ l $k_4 = 0.08$ /day/ $\mu$ l $e_{40} = 0.0002$ cells/day/ $\mu$ l $\eta_2 = 0.45$ $x_{40} = 730$ cells/ $\mu$ l	$D = 5000$ units $\sigma_0 = 0.5$ mg/day $\sigma_2 = 30$ mg/day $\sigma_3 = 5$ mg/day $\lambda_0 = 5$ mg/L $\lambda_2 = 15$ mg/L $\lambda_3 = 0.025$ mg/L $x_{50} = 1500$ cells/ $\mu$ l $k_5 = 0.0$ /day constant infusion
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**Figure 2 Simulation results using parametric configuration  $P_2$**

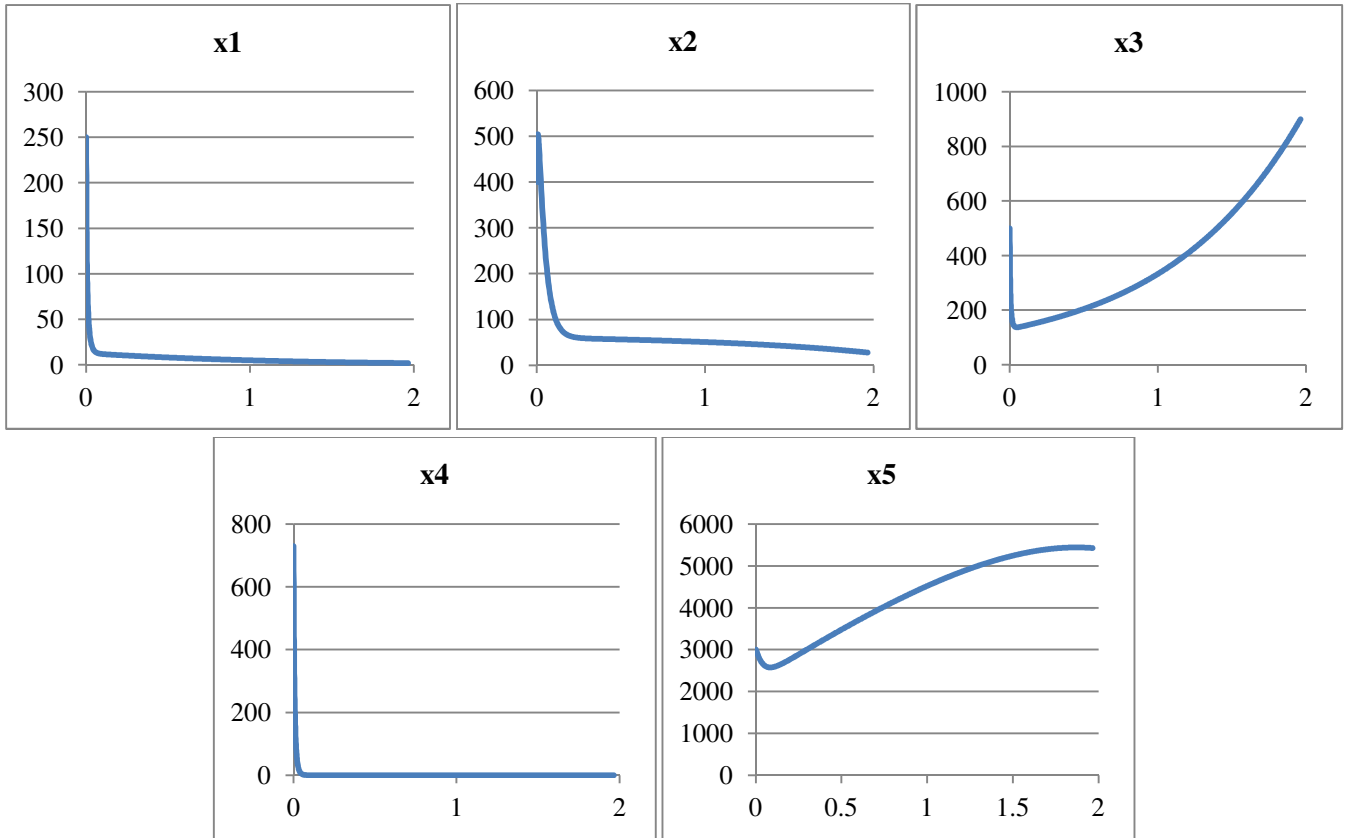
### 7.3 Simulation results for hypothetical AIDS patient #3

For this scenario, the patho-physiological parametric configuration of hypothetical patient #3 is as shown in Table 3. It must be noted that HIV-1 AIDS dynamics during HAART treatment protocol in this patient is exacerbated by contributions from latent reservoirs in contrast to the other hypothetical patients. The results of the simulation are shown in Figure 3. This is a non-equilibrium AIDS configuration simulation as it is evident parametric configuration and by the simulation time profile. The simulation results show that during the given HAART protocol, the hypothetical AIDS patient #3 experiences annihilation of uninfected CD4+ T cells and the HIV-1 specific CD8+ T cells. Consequently, immune system paralysis occurs as a consequence of the low CD4+ T cell number density, leading to an exponential increase of the blood plasma HIV-1 viremia. This simulation represents an unfavorable AIDS outcome during HAART.

**Table 3 Parametric Configuration  $P_3$**

TABLE 3. Hypothetical AIDS Patient Parametric Configuration  $P_3$

$S_1 = 800 \text{ /day/}\mu\text{l}$ $a_1 = 0.15 \text{ /day/cell/}\mu\text{l}$ $b_1 = 0.003 \text{ /cell/}\mu\text{l}$ $\alpha_1 = 0.5 \text{ /day/virion/}\mu\text{l}$ $k_1 = 0.0005 \text{ /day/}\mu\text{l}$ $q_1 =$ $0.00045 \text{ /day/}\mu\text{l/cell}$ $e_{10} = 0.0025$ $\text{cells/day/}\mu\text{l}$ $x_{10} = 250 \text{ cells/}\mu\text{l}$	$S_2 = 800 \text{ /day/}\mu\text{l}$ $a_2 = 0.03 \text{ /day/cell/}\mu\text{l}$ $b_2 = 0.004 \text{ /cell/}\mu\text{l}$ $\alpha_2 = 0.5 \text{ /day/virion/}\mu\text{l}$ $k_2 = 0.005 \text{ /day/}\mu\text{l}$ $q_2 =$ $0.00001 \text{ /day/}\mu\text{l/cell}$ $\beta_1 = 1.05$ $\text{virions/CD4}^+ \text{ /day}$ $K_1 = 0.0001 \text{ /day/}\mu\text{l}$ $e_{20} = 0.005 \text{ cells/day/}\mu\text{l}$ $\xi_2 = 0.85$ $x_{20} = 400 \text{ cells/}\mu\text{l}$	$S_3 = 10 \text{ /day/}\mu\text{l}$ $\beta_2 = 0.0015$ $\text{virions/CD4}^+ \text{ /day/}\mu\text{l}$ $\beta_3 = 1.05$ $\text{virions/CD4}^+ \text{ /day}$ $\alpha_3 = 0.027 \text{ /day/virion/}\mu\text{l}$ $k_3 = 0.0001 \text{ /day/}\mu\text{l}$ $e_{30} = 0.0001 \text{ /day}$ $\eta_1 = 0.25$ $\xi_3 = 0.001$ $x_{30} = 500 \text{ cells/}\mu\text{l}$	$S_4 = 10 \text{ /day/}\mu\text{l}$ $a_4 = 0.35 \text{ /day/cell/}\mu\text{l}$ $b_4 = 0.002 \text{ /cell/}\mu\text{l}$ $K_2 = 0.095 \text{ /day/}\mu\text{l}$ $k_4 = 0.08 \text{ /day/}\mu\text{l}$ $e_{40} = 0.0002$ $\text{cells/day/}\mu\text{l}$ $\eta_2 = 0.45$ $x_{40} = 730 \text{ cells/}\mu\text{l}$	$D = 5000 \text{ units}$ $\sigma_0 = 1.0 \text{ mg/day}$ $\sigma_2 = 30 \text{ mg/day}$ $\sigma_3 = 5 \text{ mg/day}$ $\lambda_0 = 5 \text{ mg/L}$ $\lambda_2 = 15 \text{ mg/L}$ $\lambda_3 = 0.025 \text{ mg/L}$ $x_{50} = 3000 \text{ cells/}\mu\text{l}$ $k_5 = 0.0 \text{ /day}$ $\text{Constant Infusion}$
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**Figure 3 Simulation results using parametric configuration  $P_3$**

## 8. Summary, Discussion and Future Work

The current paper is elaborate and quantitative attempt to construct medically applicable mathematic models and derive criteria for efficacious HAART protocol for an AIDS patient. The mathematical equations presented in Section 3 are a generalization of the models by Wodarz and Nowak [16], Perelson et al [13], Tang et al. [15], and many other authors by incorporating more clinically relevant parameters of HIV-1 patho-physiodynamics. In particular, interleukin-2 activation of both  $CD_4^+$  T cells and  $CD_8^+$  T cells are implicitly incorporated into the model. In this paper, it has been demonstrated that the parameters  $(a_1, b_1)$  and  $(a_4, b_4)$  play a significant role in determining the efficacious outcomes of HAART. Clearly, the clinically desirable outcomes are the rest points E4 and E3. The criteria for the existence of E4 and E3 are listed respectively in Theorem 5.1, and Theorem 5.4. These theorems give the conditions under which both E4 and E3 can be temporarily or transiently attained. The criteria for E4 and E3 to become global attractors are listed Theorem 6.1 and Theorem 6.2. These theorems, expressed in terms of simple clinically attainable and measurable parameters, give the sufficient conditions for the

cure of AIDS. It will be emphasized that there exist other therapeutic criteria but these are intractable and extremely arduous to achieve in a clinical setting.

One essential feature of this research is that the HAART protocol is implemented by using the constant continuous intravenous infusion or constant continuous transdermal infusion of the drug. As such, the criteria derived in this paper are applies to these settings. The mathematical model for intermittent caplet or matrix tablet per oral drug administration can be derived using the non-autonomous version of the model equations involving the drug input function  $f(t)$  discussed by Nani and Jin in [10].

The scenarios for clinical failure during HAART due to extreme drug toxicity are represented by the rest points E1 and E5. The criteria for these therapeutic outcomes are described by Theorem 5.3 and Theorem 5.2, respectively.

The future work will involve the construction of mathematical models which will describe an AIDS therapy using HAART in addition to ACI (Active Cellular Immunotherapy) involving interleukin-2. Also, in the future, the mathematical model for HAART will include the roles of mutations such as the delta32 CCR5 on therapeutic outcomes.

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